A Singular Integral Estimate of Brascamp-Lieb Type

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Outline

- Overview and Motivation
- Statement of the Main Theorem
- Application of the Main Theorem
- Reduction to Model Operators

Overview - Classical Brascamp-Lieb Inequalities

Definition

Classical Brascamp-Lieb inequalities refer to a class of inequalities of the following form

$$\left| \int_{\mathbb{R}^d} \prod_{j=1}^{n} F_j(L_j(x))dx \right| \leq \mathrm{BL}(\mathbf{L},\mathbf{p}) \prod_{j=1}^{n} \|F_j\|_{L^{p_j}(\mathbb{R}^{k_j})}, \tag{1}$$

where

- $\mathbf{L} := (L_j)_{j=1}^n$ and for each $1 \leq j \leq n$, $L_j : R^d \to R^{k_j}$ is a linear surjection;
- $\mathbf{p} := (p_j)_{j=1}^n$ and $p_j \geq 1$;
- $\mathrm{BL}(\mathbf{L},\mathbf{p})$ represents the smallest constant for which (1) holds for all $f_j : \mathbb{R}^{k_j} \to \mathbb{R}^+$.

References: Brascamp-Lieb, Bennett-Carbery-Christ-Tao, etc.

Overview - Classical Brascamp-Lieb Inequalities

Some well-known examples of classical Brascamp-Lieb inequalities, namely

$$\left|\int_{\mathbb{R}^d} \prod_{j=1}^{n} F_j(L_j(x))dx\right| \leq \mathrm{BL}(\mathbf{L},\mathbf{p}) \prod_{j=1}^{n} \|F_j\|_{L^{p_j}(\mathbb{R}^{k_j})},$$

include

- Hölder's inequality

$$\left|\int_{\mathbb{R}^d} \prod_{j=1}^{n} F_j(x)dx\right| \leq \prod_{j=1}^{n} \|F_j\|_{L^{p_j}(\mathbb{R}^d)}$$

  where $L_j := id$ for each j and $\sum_{j=1}^{n} \frac{1}{p_j} = 1$.

- Loomis-Whitney inequality

$$\left|\int_{\mathbb{R}^d} \prod_{j=1}^{d} F_j(\pi_j(x))dx\right| \leq \prod_{j=1}^{d} \|F_j\|_{L^{d-1}(\mathbb{R}^{d-1})}$$

  where $L_j := \pi_j$ and $\pi_j : \mathbb{R}^d \to \mathbb{R}^{d-1}$ such that

$$\pi_j(x) = (x_1, \cdots, x_{j-1}, x_{j+1}, \cdots, x_d).$$

Overview - Singular Integral of Brascamp-Lieb Type

Definition

A multilinear singular integral operator T is a *singular integral of Brascamp-Lieb type* if when all the kernels are set to be Dirac distributions δ_0, $T(F_1, \ldots, F_n)$ can be reduced to the integrand on the left hand side of a classical Brascamp-Lieb inequality.

Remark

Scaling of a singular Brascamp-Lieb inequality thus follows from its corresponding classical Brascamp-Lieb inequality.

Overview - Examples of Singular Brascamp-Lieb of Hölder Scaling

<table>
<tr><th>Classical Brascamp-Lieb</th><th>Singular Brascamp-Lieb</th><th>References</th></tr>
<tr>
<td rowspan="2">Hölder's inequality

Scaling:

$L^{p_1} \times \cdots L^{p_n} \to L^p$

with

$\sum_{i=1}^n \frac{1}{p_i} = \frac{1}{p}, 1 < p_i < \infty$

for each i</td>
<td>Coifman-Meyer multiplier
Singular integral:
$p.v. \int K(t_1, t_2) f_1(x - t_1) f_2(x - t_2) dt_1 dt_2$

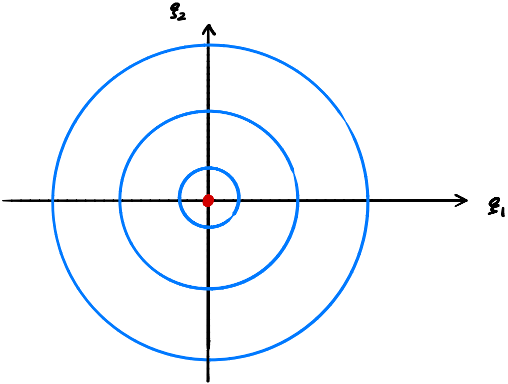
Multiplier:
$\int m(\xi_1, \xi_2) \widehat{f_1}(\xi_1) \widehat{f_2}(\xi_2) e^{2\pi i x(\xi_1 + \xi_2)} d\xi_1 d\xi_2$
where $m := \widehat{K}$.</td>
<td>Coifman-Meyer</td>
</tr>
<tr>
<td>Bi-parameter paraproduct
Singular integral:
$p.v. \int K_1(t_1, t_2) K_2(s_1, s_2) \cdot$
$f_1(x - t_1, y - s_1) f_2(x - t_2, y - s_2) dt_1 dt_2 ds_1 ds_2$

Multiplier:
$\int m_1(\xi_1, \xi_2) m_2(\eta_1, \eta_2) \widehat{f_1}(\xi_1, \eta_1) \widehat{f_2}(\xi_2, \eta_2) \cdot$
$e^{2\pi i x(\xi_1 + \xi_2)} e^{2\pi i y(\eta_1 + \eta_2)} d\xi_1 d\xi_2 d\eta_1 d\eta_2$
where $m_1 := \widehat{K_1}$, $m_2 := \widehat{K_2}$.</td>
<td>Muscalu-Pipher
-Tao-Thiele</td>
</tr>
</table>

Coifman-Meyer multiplier

$$\int m(\xi_1, \xi_2)\, \hat{f}_1(\xi_1)\, \hat{f}_2(\xi_2)\, e^{2\pi i x(\xi_1+\xi_2)}\, d\xi_1\, d\xi_2$$

where $|\partial^\alpha m(\xi_1, \xi_2)| \lesssim \dfrac{1}{|(\xi_1, \xi_2)|^{|\alpha|}}$

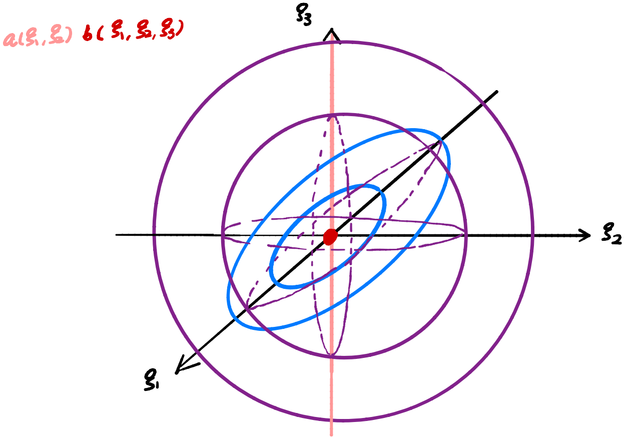


Overview - Examples of Singular Brascamp-Lieb of Hölder Scaling

Classical Brascamp-Lieb	Singular Brascamp-Lieb	References
Hölder's Inequality Scaling: $L^{p_1} \times \cdots L^{p_n} \to L^p$ with $\sum_{i=1}^n \frac{1}{p_i} = \frac{1}{p}, 1 < p_i < \infty$ for each i	**Flag paraproduct** Singular integral: $p.v. \int K(t_1, t_2)\tilde{K}(s_1, s_2) \cdot f_1(x - t_1) f_2(x - t_2 - s_1) f(x - s_2) dt_1 dt_2 ds_1 ds_2$ Multiplier: $\int a(\xi_1, \xi_2) b(\xi_1, \xi_2, \xi_3) \widehat{f_1}(\xi_1)\widehat{f_2}(\xi_2)\widehat{f_3}(\xi_3) \cdot e^{2\pi i x(\xi_1+\xi_2)} d\xi_1 d\xi_2 d\xi_3$ where $a := \widehat{K}$, $b := \widehat{\tilde{K}}$.	Muscalu Miyachi-Tomita
	Twisted paraproduct Singular integral: $p.v. \int K(t, s) \cdot f_1(x - t, y) f_2(x, y - s) dt ds$ Multiplier: $\int m(\xi_1, \eta_2)\widehat{f_1}(\xi_1, \eta_1)\widehat{f_2}(\xi_2, \eta_2) \cdot e^{2\pi i x(\xi_1+\xi_2)} e^{2\pi i y(\eta_1+\eta_2)} d\xi_1 d\xi_2 d\eta_1 d\eta_2$ where $m := \widehat{K}$.	Kovač Durcik-Thiele

Flag paraproduct

$$\int a(\xi_1,\xi_2)\, b(\xi_1,\xi_2,\xi_3)\, \hat{f}_1(\xi_1)\, \hat{f}_2(\xi_3)\, \hat{f}_3(\xi_3)\, e^{2\pi i x(\xi_1+\xi_2+\xi_3)}\, d\xi_1 d\xi_2 d\xi_3$$



Overview - Examples of Singular Brascamp-Lieb of non-Hölder Scaling

Classical Brascamp-Lieb	Singular Brascamp-Lieb	References
Loomis-Whitney inequality: $\left(\int_{\mathbb{R}^d} \lvert\prod_{j=1}^{d} f_j(\pi_j(x))\rvert^r dx\right)^{\frac{1}{r}}$ $\leq \prod_{j=1}^{d} \lVert f_j\rVert_{L^{r(d-1)}(\mathbb{R}^{d-1})}$ Scaling: $L^{r(d-1)}(\mathbb{R}^{d-1}) \times \cdots L^{r(d-1)}(\mathbb{R}^{d-1}) \to L^r$ with $r > 0$ and $r(d-1) > 1$ Alternatively, Hölder scaling with mixed norms: $L^{\infty}_{x_1}(L^{r(d-1)}_{x_2,\ldots,x_d}) \times \cdots L^{r(d-1)}_{x_1,\ldots,x_{d-1}}(L^{\infty}_{x_d})$ $\to L^r$	**Coifman-Meyer multiplier** Singular integral: $p.v. \int_{\mathbb{R}^{d^2}} K(t^1,\cdots,t^d)\cdot$ $f_1(x_2 - t_1^2,\cdots,x_d - t_1^d)$ $\vdots$ $f_d(x_1 - t_d^1,\cdots,x_{d-1} - t_d^{d-1}) dt^1\cdots dt^d$ Multiplier: $\int m(\xi^1,\xi^2,\cdots,\xi^{d-1},\xi^d)\cdot$ $\widehat{f}_1(\quad \xi_1^2,\cdots,\xi_1^{d-1},\xi_1^d)$ $\vdots$ $\widehat{f}_d(\xi_d^1,\xi_d^2,\cdots,\xi_d^{d-1}\quad)$ $e^{2\pi i x_1(\xi_2^1+\cdots+\xi_d^1)} e^{2\pi i x_2(\xi_1^2+\cdots+\xi_d^2)}$ $e^{2\pi i x_{d-1}(\xi_1^{d-1}+\cdots+\xi_d^{d-1})} e^{2\pi i x_d(\xi_1^d+\cdots+\xi_{d-1}^d)} d\xi^1\cdots d\xi^d$ where $t^i := (t_1^i,\cdots,t_d^i) \in \mathbb{R}^d$, $\xi^i := (\xi_1^i,\cdots,\xi_d^i) \in \mathbb{R}^d$ and $m := \widehat{K}$.	Benea-Muscalu

Overview - Examples Singular Brascamp-Lieb of non-Hölder Scaling

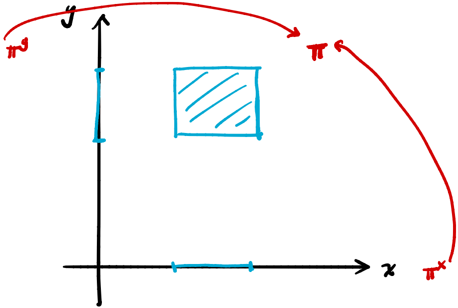
Classical Brascamp-Lieb	Singular Brascamp-Lieb	References
Loomis-Whitney inequality:	**Multiparameter paraproduct**	$d = 2$:
$\left(\int_{\mathbb{R}^d} \| \prod_{j=1}^{d} f_j(\pi_j(x)) \|^r dx \right)^{\frac{1}{r}}$ $\leq \prod_{j=1}^{d} \|f_j\|_{L^{r(d-1)}(\mathbb{R}^{d-1})}$	Singular integral: $p.v. \int_{\mathbb{R}^{d^2}} \prod_{j=1}^{d} K_j(t^j) \cdot$ $f_1(x_2 - t_1^2, \cdots, x_d - t_1^d)$ $\vdots$ $f_d(x_1 - t_d^1, \cdots, x_{d-1} - t_d^{d-1}) dt^1 \cdots dt^d$	Benea-Muscalu
Scaling: $L^{r(d-1)}(\mathbb{R}^{d-1}) \times \cdots L^{r(d-1)}(\mathbb{R}^{d-1}) \to L^r$ with $r > 0$ and $r(d-1) > 1$	Multiplier:	$d \geq 2$:
Alternatively, Hölder scaling with mixed norms: $L^{\infty}_{x_1}(L^{r(d-1)}_{x_2,\ldots,x_d}) \times \cdots L^{r(d-1)}_{x_1,\ldots,x_{d-1}}(L^{\infty}_{x_d}) \to L^r$	$\int \prod_{j=1}^{n} m_j(\xi^j) \cdot$ $\widehat{f}_1(\xi_1^2, \cdots, \xi_1^{d-1}, \xi_1^d)$ $\vdots$ $\widehat{f}_d(\xi_d^1, \xi_d^2, \cdots, \xi_d^{d-1})$ $e^{2\pi i x_1(\xi_2^1 + \cdots + \xi_d^1)} e^{2\pi i x_2(\xi_1^2 + \cdots + \xi_d^2)}$ $e^{2\pi i x_{d-1}(\xi_1^{d-1} + \cdots + \xi_d^{d-1})} e^{2\pi i x_d(\xi_1^d + \cdots + \xi_{d-1}^d)} d\xi^1 \cdots d\xi^d$ where $t^i := (t_1^i, \cdots, t_d^i) \in \mathbb{R}^d$, $\xi^i := (\xi_1^i, \cdots, \xi_d^i) \in \mathbb{R}^d$ and $m_j := \widehat{K_j}$.	Muscalu-Z., Unpublished notes.

Classical Brascamp-Lieb	Singular Brascamp-Lieb	References
Loomis-Whitney inequality: $\left(\int_{\mathbb{R}^d} \lvert \prod_{j=1}^{d} f_j(\pi_j(x)) \rvert^r dx\right)^{\frac{1}{r}}$ $\leq \prod_{j=1}^{d} \lVert f_j \rVert_{L^{r(d-1)}(\mathbb{R}^{d-1})}$ Scaling: $L^{r(d-1)}(\mathbb{R}^{d-1}) \times \cdots L^{r(d-1)}(\mathbb{R}^{d-1})$ $\to L^r$ with $r > 0$ and $r(d-1) > 1$ Alternatively, Hölder scaling with mixed norms: $L^\infty_{x_1}(L^{r(d-1)}_{x_2,\dots,x_d}) \times \cdots L^{r(d-1)}_{x_1,\dots,x_{d-1}}(L^\infty_{x_d})$ $\to L^r$	**Multiparameter paraproduct** Singular integral: $p.v. \int_{\mathbb{R}^{d^2}} \prod_{j=1}^{d} K_j(t^j)\cdot$ $f_1(x_2 - t_1^2, \cdots, x_d - t_1^d)$ $\vdots$ $f_d(x_1 - t_d^1, \cdots, x_{d-1} - t_d^{d-1}) dt^1 \cdots dt^d$ Multiplier: $\int \tilde{m}_1(\xi_2^1, \cdots, \xi_d^1)\tilde{m}_2(\xi_1^2, \cdots, \xi_d^2)\cdots$ $\tilde{m}_{d-1}(\xi_1^{d-1}, \cdots, \xi_d^{d-1})\tilde{m}_d(\xi_1^d, \cdots, \xi_d^{d-1})$ $\widehat{f}_1(\quad \xi_1^2, \cdots, \xi_1^{d-1}, \xi_1^d)$ $\vdots$ $\widehat{f}_d(\xi_d^1, \xi_d^2, \cdots, \xi_d^{d-1} \quad)$ $e^{2\pi i x_1(\xi_2^1+\cdots+\xi_d^1)} e^{2\pi i x_2(\xi_1^2+\cdots+\xi_d^2)}$ $e^{2\pi i x_{d-1}(\xi_1^{d-1}+\cdots+\xi_d^{d-1})} e^{2\pi i x_d(\xi_1^d+\cdots+\xi_{d-1}^d)} d\xi^1 \cdots d\xi^d$ where $t^i := (t_1^i, \cdots, t_d^i) \in \mathbb{R}^d$, $\xi^i := (\xi_1^i, \cdots, \xi_d^i) \in \mathbb{R}^d$ and $\tilde{m}_1(\xi^1) := m_1(0, \xi_2^1, \cdots, \xi_d^1)$ etc.	$d = 2$: Benea-Muscalu $d \geq 2$: Muscalu-Z., Unpublished notes.

Classical B-L

2-D Loomis Whitney: $\int |f(x)g(y)| \leq \int |f(x)|\,dx \int |g(y)|\,dy$

$1_F \otimes 1_G$



Classical Brascamp-Lieb	Singular Brascamp-Lieb	References
2D Loomis-Whitney inequality + Hölder $\left(\int \lvert f_1(x)f_2(x)g_1(y)g_2(y)h(x,y)\rvert^r\,dxdy\right)^{\frac{1}{r}}$ $\leq \lVert f_1\rVert_{L^{p_1}(\mathbb{R})}\lVert f_2\rVert_{L^{q_1}(\mathbb{R})}\cdot$ $\lVert g_1\rVert_{L^{p_2}(\mathbb{R})}\lVert g_2\rVert_{L^{q_2}(\mathbb{R})}\lVert h\rVert_{L^s(\mathbb{R}^2)}$	**5 -linear singular integral operator** $\int_{\mathbb{R}^2} a_1(\xi_1,\xi_2)b_1(\xi_1,\xi_2,\xi_3)a_2(\eta_1,\eta_2)b_2(\eta_1,\eta_2,\eta_3)\cdot$ $\widehat{f_1}(\xi_1)\widehat{f_2}(\xi_2)\hat{g_1}(\eta_1)\hat{g_2}(\eta_2)\hat{h}(\xi_3,\eta_3)$ $e^{2\pi ix(\xi_1+\xi_2+\xi_3)}e^{2\pi iy(\eta_1+\eta_2+\eta_3)}d\xi_1 d\xi_2 d\xi_3 d\eta_1 d\eta_2 d\eta_3$	Pipher-Lu -Zhang Muscalu-Z.
Scaling: $L^{p_1}(\mathbb{R})\times L^{q_1}(\mathbb{R})\times L^{p_2}(\mathbb{R})\times L^{q_2}(\mathbb{R})\times L^s(\mathbb{R}^2)\to L^r$ with $\frac{1}{p_1}+\frac{1}{q_1}=\frac{1}{p_2}+\frac{1}{q_2}=\frac{1}{r}-\frac{1}{s}$, $1\leq p_1,p_2,q_1,q_2\leq\infty$		

Main Theorem - Statement

Suppose $a_1, a_2 \in L^\infty(\mathbb{R}^2)$, $b_1, b_2 \in L^\infty(\mathbb{R}^3)$, where a_1, a_2 are smooth away from from $\{(\xi_1,\xi_2)=0\}$,$\{(\eta_1,\eta_2)=0\}$ respectively and b_1, b_2 are smooth away from $\{(\xi_1,\xi_2,\xi_3)=0\}$, $\{(\eta_1,\eta_2,\eta_3)=0\}$ respectively. Assume they satisfy the following Marcinkiewicz-Mikhlin-Hörmander conditions:

$$\left|\partial^{\alpha_1} a_1(\xi_1,\xi_2)\right| \lesssim \frac{1}{|(\xi_1,\xi_2)|^{|\alpha_1|}}, \left|\partial^{\alpha_2} a_2(\eta_1,\eta_2)\right| \lesssim \frac{1}{|(\eta_1,\eta_2)|^{|\alpha_2|}},$$

$$\left|\partial^{\beta_1} b_1(\xi_1,\xi_2,\xi_3)\right| \lesssim \frac{1}{|(\xi_1,\xi_2,\xi_3)|^{|\beta_1|}}, \left|\partial^{\beta_2} b_2(\eta_1,\eta_2,\eta_3)\right| \lesssim \frac{1}{|(\eta_1,\eta_2,\eta_3)|^{|\beta_2|}},$$

for sufficiently many multi-indices $\alpha_1, \alpha_2, \beta_1, \beta_2$. For $f_1, f_2, g_1, g_2 \in \mathcal{S}(\mathbb{R})$ and $h \in \mathcal{S}(\mathbb{R}^2)$ where $\mathcal{S}(\mathbb{R})$ and $\mathcal{S}(\mathbb{R}^2)$ denote the Schwartz spaces, define a five-linear operator as

$$\begin{aligned} T_{a_1 b_1 a_2 b_2}(f_1, f_2, g_1, g_2, h)(x,y) := \int_{\mathbb{R}^6} & a_1(\xi_1,\xi_2) b_1(\xi_1,\xi_2,\xi_3) a_2(\eta_1,\eta_2) b_2(\eta_1,\eta_2,\eta_3) \\ & \hat{f}_1(\xi_1)\hat{f}_2(\xi_2)\hat{g}_1(\eta_1)\hat{g}_2(\eta_2)\hat{h}(\xi_3,\eta_3) \\ & e^{2\pi i x(\xi_1+\xi_2+\xi_3)} e^{2\pi i y(\eta_1+\eta_2+\eta_3)} d\xi_1 d\xi_2 d\xi_3 d\eta_1 d\eta_2 d\eta_3. \end{aligned} \quad (2)$$

Theorem

$$T_{a_1 b_1 a_2 b_2} : L^{p_1}(\mathbb{R}) \times L^{q_1}(\mathbb{R}) \times L^{p_2}(\mathbb{R}) \times L^{q_2}(\mathbb{R}) \times L^{s}(\mathbb{R}^2) \to L^{r}(\mathbb{R}^2),$$

for

- $1 < p_1, p_2, q_1, q_2, s \leq \infty, r > 0$;
- $0 < \frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2} = \frac{1}{r} - \frac{1}{s}$.

Corollary - Leibniz Rule

- One corollary of the theorem is a Leibniz rule which captures the nonlinear interaction of waves coming from transversal directions. In general, Leibniz rules refer to inequalities involving norms of derivatives.
- For $\alpha \geq 0$ and $f \in \mathcal{S}(\mathbb{R}^d)$ a Schwartz function in $\mathbb{R}^d$, define the homogeneous derivative of f as

$$D^\alpha f := \mathcal{F}^{-1}\left(|\xi|^\alpha \widehat{f}(\xi)\right).$$

Corollary

Suppose $f_1, f_2 \in \mathcal{S}(\mathbb{R})$, $g_1, g_2 \in \mathcal{S}(\mathbb{R})$ and $h \in \mathcal{S}(\mathbb{R}^2)$. Then for $\beta_1, \beta_2, \alpha_1, \alpha_2 > 0$ sufficiently large and $1 < p_1^j, p_2^j, q_1^j, q_2^j, s^j \leq \infty$, $r > 0$, $(p_1^j, q_1^j), (p_2^j, q_2^j) \neq (\infty, \infty)$, $\frac{1}{p_1^j} + \frac{1}{q_1^j} = \frac{1}{p_2^j} + \frac{1}{q_2^j} = \frac{1}{r} - \frac{1}{s^j}$ for each $j = 1, \ldots, 16$,

$$\begin{aligned}
&\|D_1^{\beta_1} D_2^{\beta_2}(D_1^{\alpha_1} D_2^{\alpha_2}(f_1^x f_2^x g_1^y g_2^y) h^{x,y})\|_{L^r(\mathbb{R}^2)} \\
\lesssim\ & \text{sum of } 16 \text{ terms of the forms:} \\
&\|D_1^{\alpha_1+\beta_1} f_1\|_{L^{p_1^1}(\mathbb{R})} \|f_2\|_{L^{q_1^1}(\mathbb{R})} \|D_2^{\alpha_2+\beta_2} g_1\|_{L^{p_2^1}(\mathbb{R})} \|g_2\|_{L^{q_2^1}(\mathbb{R})} \|h\|_{L^{s^1}(\mathbb{R}^2)} + \\
&\|f_1\|_{L^{p_1^2}(\mathbb{R})} \|D_1^{\alpha+\beta_1} f_2\|_{L^{q_1^2}(\mathbb{R})} \|D_2^{\alpha_2+\beta_2} g_1\|_{L^{p_2^2}(\mathbb{R})} \|g_2\|_{L^{q_2^2}(\mathbb{R})} \|h\|_{L^{s^2}(\mathbb{R}^2)} + \\
&\|D_1^{\alpha+\beta_1} f_1\|_{L^{p_1^3}(\mathbb{R})} \|f_2\|_{L^{q_1^3}(\mathbb{R})} \|D_2^{\alpha_2} g_1\|_{L^{p_2^3}(\mathbb{R})} \|g_2\|_{L^{q_2^3}(\mathbb{R})} \|D_2^{\beta_2} h\|_{L^{s^3}(\mathbb{R}^2)} + \ldots
\end{aligned}$$

Reduction to Model Operators - Littlewood-Paley Decomposition

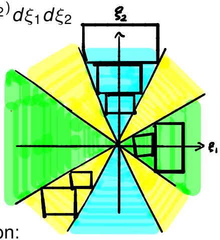
- Recall Leibniz rule corresponding to the boundedness of Coifman-Meyer multiplier:

$$\|D^\alpha(f_1 f_2)\|_r \lesssim \|D^\alpha f_1\|_{p_1}\|f_2\|_{q_1} + \|f_1\|_{p_2}\|D^\alpha f_2\|_{q_2}$$

- $D^\alpha(f_1 f_2)(x) := \mathcal{F}^{-1}(|\xi|^\alpha \hat{f}_1 * \hat{f}_2(\xi))(x) = \int |\xi_1+\xi_2|^\alpha \hat{f}_1(\xi_1)\hat{f}_2(\xi_2)e^{2\pi i x(\xi_1+\xi_2)}d\xi_1 d\xi_2$

- We want to separate the domain of integration into the following three regions:

(1) $|\xi_1+\xi_2| \sim |\xi_2|$
(2) $|\xi_1+\xi_2| \sim |\xi_1|$
(3) $|\xi_1| \sim |\xi_2|$



- We perform the Littlewood-Paley decomposition to achieve the above separation:
Recall $1 = \sum_{k\in\mathbb{Z}} \widehat{\psi}_k(\xi)$, where $\widehat{\psi}_k$ are bump functions with $\text{supp}(\widehat{\psi}_k) \subseteq \{\xi : 2^{k-1} \lesssim |\xi| \lesssim 2^k\}$.
For our purpose,

$$1(\xi_1,\xi_2) = \sum_{k_1,k_2\in\mathbb{Z}} \hat{\psi}_k(\xi_1)\hat{\psi}_{k_2}(\xi_2)$$

$$= \sum_{k_2} \hat{\psi}_{k_2}(\xi_2) \underbrace{\sum_{k_1:k_1\leq k_2-100} \hat{\psi}_{k_1}(\xi_1)}_{\hat{\varphi}_{k_2}(\xi_1)} + \sum_{k_1} \hat{\psi}_{k_1}(\xi_1) \underbrace{\sum_{k_2:k_2\leq k_1-100} \hat{\psi}_{k_2}(\xi_2)}_{\hat{\varphi}_{k_1}(\xi_2)} + \sum_{k_1} \hat{\psi}_{k_1}(\xi_1)\hat{\tilde{\psi}}_{k_1}(\xi_2).$$

- We introduce the notation that a bump function is of ψ-type if its Fourier support is away from 0 and of φ-type otherwise.
- With the separation, we have

$$
\begin{aligned}
D^{\alpha}(f_1 f_2)(x) = &\int \sum_k \hat{\varphi}_k(\xi_1)\hat{\psi}_k(\xi_2)\frac{|\xi_1+\xi_2|^{\alpha}}{|\xi_2|^{\alpha}}\hat{f}(\xi_1)|\xi_2|^{\alpha}\hat{g}(\xi_2)e^{2\pi i x(\xi_1+\xi_2)}d\xi_1 d\xi_2 + \\
&\int \sum_k \hat{\psi}_k(\xi_1)\hat{\varphi}_k(\xi_2)\frac{|\xi_1+\xi_2|^{\alpha}}{|\xi_1|^{\alpha}}|\xi_1|^{\alpha}\hat{f}(\xi_1)\hat{g}(\xi_2)e^{2\pi i x(\xi_1+\xi_2)}d\xi_1 d\xi_2 + \\
&\int \sum_k \hat{\psi}_k(\xi_1)\hat{\tilde{\psi}}_k(\xi_2)|\xi_1+\xi_2|^{\alpha}\hat{f}(\xi_1)\hat{g}(\xi_2)e^{2\pi i x(\xi_1+\xi_2)}d\xi_1 d\xi_2
\end{aligned}
$$

- We introduce the notation that a bump function is of ψ-type if its Fourier support is away from 0 and of φ-type otherwise.
- With the separation, we have

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$$\int \sum_k \hat{\psi}_k(\xi_1)\hat{\varphi}_k(\xi_2)\frac{|\xi_1+\xi_2|^\alpha}{|\xi_1|^\alpha}\widehat{D^\alpha f}(\xi_1)\hat{g}(\xi_2)e^{2\pi i x(\xi_1+\xi_2)}d\xi_1 d\xi_2 +$$
$$\int \sum_k \hat{\psi}_k(\xi_1)\hat{\tilde{\psi}}_k(\xi_2)|\xi_1+\xi_2|^\alpha\hat{f}(\xi_1)\hat{g}(\xi_2)e^{2\pi i x(\xi_1+\xi_2)}d\xi_1 d\xi_2$$

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\end{aligned}
$$

- In general, for $m(\vec{\xi})$ satisfying the Marcinkiewicz-Mikhlin-Hörmander condition, namely

$$
|\partial^\alpha m(\vec{\xi})| \lesssim \frac{1}{|\vec{\xi}|^{|\alpha|}}
$$

  for sufficiently many multi-indices α, m restricted to the Whitney cube with respect to the origin is essentially constant.

- By applying the Littlewood-Paley decomposition to a_1 in the frequency plane and b_1 in the frequency space,

$$a_1(\xi_1,\xi_2) = \sum_k \underbrace{\widehat{\phi}_k^1(\xi_1)\widehat{\phi}_k^2(\xi_2)}_{\text{at least one of } \psi\text{-type}}$$

and

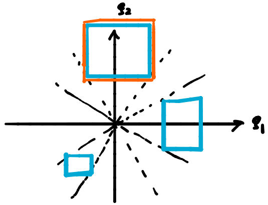
$$b_1(\xi_1,\xi_2,\xi_3) = \sum_i \underbrace{\widehat{\phi}_i^1(\xi_1)\widehat{\phi}_i^2(\xi_2)\widehat{\phi}_i^3(\xi_3)}_{\text{at least one of } \psi\text{-type}}$$

- The symbol $a_1(\xi_1,\xi_2)b_1(\xi_1,\xi_2,\xi_3)$ can be rewritten as a sum of 2 types of symbols:

(1) $m_1^1(\xi_1,\xi_2,\xi_3) := \sum_{k,i} \widehat{\varphi}_k^1(\xi_1)\widehat{\psi}_k^2(\xi_2)\widehat{\phi}_i^1(\xi_1)\widehat{\phi}_i^2(\xi_2)\widehat{\varphi}_i^3(\xi_3)$

(2) $m_1^2(\xi_1,\xi_2,\xi_3) := \sum_{k,i} \widehat{\varphi}_k^1(\xi_1)\widehat{\psi}_k^2(\xi_2)\widehat{\phi}_i^1(\xi_1)\widehat{\phi}_i^2(\xi_2)\widehat{\psi}_i^3(\xi_3)$

(1)



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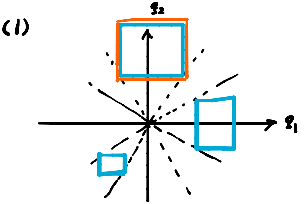

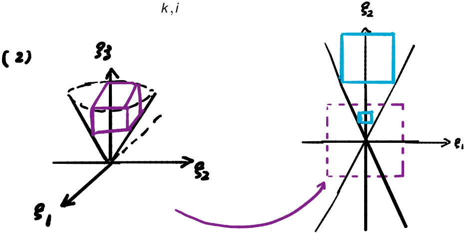

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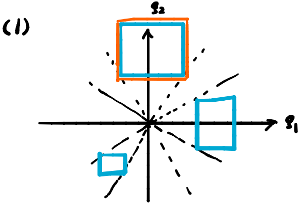
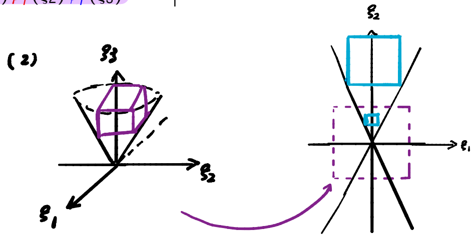
  (2) $m_1^2(\xi_1,\xi_2,\xi_3) = \sum_{k \lesssim i} \widehat{\varphi}_k^1(\xi_1)\widehat{\psi}_k^2(\xi_2)\widehat{\varphi}_i^1(\xi_1)\widehat{\varphi}_i^2(\xi_2)\widehat{\psi}_i^3(\xi_3) =: m_1^{\text{flag}}$





Reduction to Model Operators

The symbol $a_1(\xi_1, \xi_2) b_1(\xi_1, \xi_2, \xi_3) a_2(\eta_1, \eta_2) b_2(\eta_1, \eta_2, \eta_3)$ can be decomposed into 4 types of symbols:

$$
\begin{aligned}
&(m_1^{\text{paraproduct}} + m_1^{\text{flag}}) \otimes (m_2^{\text{paraproduct}} + m_2^{\text{flag}}) \\
= &\underbrace{m_1^{\text{paraproduct}} \otimes m_2^{\text{paraproduct}}}_{\text{bi-parameter paraproduct}} + m_1^{\text{paraproduct}} \otimes m_2^{\text{flag}} + m_1^{\text{flag}} \otimes m_2^{\text{paraproduct}} + m_1^{\text{flag}} \otimes m_2^{\text{flag}}.
\end{aligned}
$$