On some partial orders on $\mathcal{B}(\mathcal{H})$

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Contents

Example 1.1

Let $\mathbb{C}^n$ be the n dimensional complex (Hilbert) Euclid space and let M_n be the algebra of all complex $n \times n$ matrices. Let M_i(resp. N_i) be the m_i(resp. n_i) dimensional subspace of $\mathbb{C}^n$ for $i = 1, 2$ such that $n = m_1 + m_2 = n_1 + n_2$ and

$$\mathbb{C}^n = M_1 \oplus M_2 = N_1 \oplus N_2. \tag{1}$$

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$$\mathbb{C}^n = M_1 \oplus M_2 = N_1 \oplus N_2. \tag{1}$$

Let A, $B \in M_n$. If there are $n_i \times m_i$ matrices $B_i (i = 1, 2)$such that

$$A = \begin{pmatrix} B_1 & 0 \\ 0 & 0 \end{pmatrix}, B = \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix}$$

with respect to the decomposition (1), then we say that $A \leq B$.

1. Partial orders on $\mathcal{B}(\mathcal{H})$

It is known that "$\leq$" is a partial order in M_n.

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We now may consider this partial order in the algebra $\mathcal{B}(\mathcal{H})$(or a von Neumann algebra $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$) of all bounded linear operators on a complex Hilbert space $\mathcal{H}$.

Definition 1.2

Let $A, B \in \mathcal{M}$. If $A^*A = A^*B$, $AA^* = BA^*$, then we say that $A \overset{*}{\leq} B$.

The order $\overset{*}{\leq}$ is called the star partial order in $\mathcal{M}$.

Let M_i and $N_i (i = 1, 2)$ be closed subspaces of $\mathcal{H}$ such that

$$\mathcal{H} = M_1 \oplus M_2 = N_1 \oplus N_2. \tag{2}$$

For any $T \in \mathcal{B}(\mathcal{H})$, there are $T_{ji} \in \mathcal{B}(M_i, N_j)(i, j = 1, 2)$ such that

$$T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \tag{3}$$

with respect to the orthogonal decompositions (2).

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with respect to the orthogonal decompositions (2).

Proposition 1.3

Let $A, B \in \mathcal{B}(\mathcal{H})$. Then $A \overset{*}{\leq} B$ if and only if there is a orthogonal direct decomposition (2) of $\mathcal{H}$ such that

$$A = \begin{pmatrix} A_{11} & 0 \\ 0 & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} A_{11} & 0 \\ 0 & B_{22} \end{pmatrix}. \tag{4}$$

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Definition 1.4 (J. K. Baksalary, J. Hauke,1990)

For $A,\ B \in \mathcal{B}(\mathcal{H})$, we say $A \leq^{\diamond} B$ if $\overline{R(A)} \subseteq \overline{R(B)}, \overline{R(A^*)} \subseteq \overline{R(B^*)}$ and $AA^*A = AB^*A$.

It is known that $\leq^{\diamond}$ is a partial order on $\mathcal{B}(\mathcal{H})$ and is called the diamond partial order.

Proposition 1.5

Let $A, B \in \mathcal{B}(\mathcal{H})$ and $U \in \mathcal{B}(\mathcal{H})$ is unitary.

1. If $A \overset{*}{\leq} B$(resp. $A \leq^{\diamond} B$) , then $UA \overset{*}{\leq} UB$(resp. $UA \leq^{\diamond} UB$) and $AU \overset{*}{\leq} BU$(resp. $AU \leq^{\diamond} BU$).
2. $A \overset{*}{\leq} B \Longleftrightarrow A = P_{\overline{R(A)}}B = BP_{\overline{R(A^*)}}$.
3. $A \leq^{\diamond} B \Longleftrightarrow A = P_{\overline{R(A)}}BP_{\overline{R(A^*)}} = P_{\overline{R(B)}}AP_{\overline{R(B^*)}}$.
4. If $A \overset{*}{\leq} B$, then $A \leq^{\diamond} B$.

Definition 2.1

Let " $\leq$ " be a partial order on a von Neumann algebra $\mathcal{M}$ and $\mathfrak{A} \subseteq \mathcal{M}$ a subspace. For any $A \in \mathcal{M}$ and $B \in \mathfrak{A}$, if $A \in \mathfrak{A}$ whenever $A \leq B$, then we say that $\mathfrak{A}$ is a hereditary subspace with respect to the partial order " $\leq$ ".

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If $\mathfrak{A}$ is a hereditary subspace with respect to the star(resp. diamond) partial order, then we say that $\mathfrak{A}$ is a star(resp. diamond) partial order hereditary subspace of $\mathcal{M}$.

Remark

1. $\mathfrak{A}$: star(resp. diamond) partial order hereditary $\Longrightarrow$ $\mathfrak{A}^* = \{X^* : X \in \mathfrak{A}\}$ is.
2. $\mathfrak{A}$: diamond partial order hereditary$\Longrightarrow$ $\mathfrak{A}$: star partial order hereditary.
3. $I \subseteq \mathcal{M}$: left(resp. right) ideal$\Longrightarrow$ I: diamond(star) partial order hereditary.

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We recall that if I is a weak* closed left, right or two-sided ideal in $\mathcal{M}$ respectively, then there are projections $E, F \in \mathcal{M}$ or a central projection $P \in Z(\mathcal{M}) = \mathcal{M} \cap \mathcal{M}'$ such that $I = \mathcal{M}E$, $I = F\mathcal{M}$ or $I = P\mathcal{M}$.

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In general, let $E, F \in \mathcal{M}$ be two projections. Then $E\mathcal{M}F$ is weak* closed diamond as well as star partial order-hereditary.

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Theorem 2.2

Let $\mathfrak{A}$ be a nonzero norm closed star partial order hereditary subspace in $\mathcal{B}(\mathcal{H})$. Then there exists a unique pair of projections $E, F \in \mathcal{B}(\mathcal{H})$ such that

$$\mathfrak{A} \cap \mathcal{K}(\mathcal{H}) = E\mathcal{K}(\mathcal{H})F \text{ and } \overline{\mathfrak{A}}^{w^*} = E\mathcal{B}(\mathcal{H})F,$$

where $\overline{\mathfrak{A}}^{w^*}$ is the weak* closure of $\mathfrak{A}$. That is,

$$E\mathcal{K}(\mathcal{H})F \subseteq \mathfrak{A} \subseteq E\mathcal{B}(\mathcal{H})F.$$

Example 2.3

Let $\mathcal{N}$ be an infinite dimensional Hilbert space and $\mathcal{H} = \mathcal{N} \oplus \mathcal{N}$. Put

$$\mathfrak{A} = \left\{ \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} : X_{11}, X_{12} \in \mathcal{B}(\mathcal{N}), X_{21}, X_{22} \in \mathcal{K}(\mathcal{N}) \right\}.$$

Then $\mathfrak{A}$ is a norm closed star partial order hereditary subspace such that $\mathcal{K}(\mathcal{H}) \subsetneqq \mathfrak{A} \subsetneqq \mathcal{B}(\mathcal{H})$.

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Then $\mathfrak{A}$ is a norm closed star partial order hereditary subspace such that $\mathcal{K}(\mathcal{H}) \subsetneqq \mathfrak{A} \subsetneqq \mathcal{B}(\mathcal{H})$.

It is also known that both $\mathfrak{A}^*$ and $\mathfrak{A} \cap \mathfrak{A}^*$ are also star partial order hereditary subspaces containing $\mathcal{K}(\mathcal{H})$. However $\mathfrak{A} \bigvee \mathfrak{A}^*$ is not star partial order hereditary.

Let $\mathcal{M}$ be a von Neumann algebra and $Z(\mathcal{M}) = \mathcal{M} \cap \mathcal{M}'$ the center of $\mathcal{M}$. Let $A \in \mathcal{M}$.

$$C_A = inf\{E \in Z(\mathcal{M}) : E \text{ is a projection such that } EA = A\}$$

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Theorem 2.4

Let $\mathfrak{A} \subseteq \mathcal{M}$ be a weak* closed star partial order hereditary subspace. Then there exists a unique pair of projections $E, F \in \mathcal{M}$ with the same central carriers $C_E = C_F$ such that $\mathfrak{A} = E\mathcal{M}F$.

Corollary 2.5

Let $\mathfrak{A} \subseteq \mathcal{M}$ be a weak* closed diamond partial order hereditary subspace. Then there exists a unique pair of projections $E, F \in \mathcal{M}$ with the same central carriers $C_E = C_F$ such that $\mathfrak{A} = E\mathcal{M}F$.

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Remark

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If it is also weak* closed, then it is.

We consider the unit interval with respect to a partial order.

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Put $\Sigma = \{T \in \mathcal{B}(\mathcal{H}) : 0 \overset{*}{\leq} T \overset{*}{\leq} I\}$, the unit interval with respect to the star partial order. Then

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and for any $E, F \in \Sigma$, $E \overset{*}{\leq} F \Longleftrightarrow E \leq F$.

In this case, Σ is just the lattice of subspaces in $\mathcal{H}$.

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Theorem 3.1 (A basic theorem)

Assume $\dim \mathcal{H} = n \geq 3$, that is, $\mathcal{H} = \mathbb{C}^n$. Let φ be a lattice automorphism on Σ. Then there are a ring automorphism τ on $\mathbb{C}$ and a τ linear bijection $S(S(ax+by) = \tau(a)x + \tau(b)y$, $\forall a, b \in \mathbb{C}$, $x, y \in \mathbb{C}^n$) on $\mathbb{C}^n$ such that

$$\varphi(E) = P_{R(SES^{-1})} = P_{R(SE)}, \quad \forall E \in \Sigma.$$

Theorem 3.2 (Fillmore and Longstaff)

Assume $\dim \mathcal{H} = \infty$. Let φ be a lattice automorphism on Σ. Then there is a bounded invertible linear or conjugate linear operator S on $\mathcal{H}$ such that

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Theorems 3.1 and 3.2 give a complete description of order automorphism on Σ.

Put $\Lambda = \{T \in \mathcal{B}(\mathcal{H}) : 0 \leq^{\diamond} T \leq^{\diamond} I\}$. Then Λ is a poset but not a lattice.

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Theorem 3.3

$\Lambda = \{EF : E, F \in \Sigma\}$.

Let $T \in \Lambda$. Then

$$T = P_{\overline{R(T)}} P_{N(T)^{\perp}} = P_{\overline{R(T)}} P_{\overline{R(T^*)}} = P_{N(T^*)^{\perp}} P_{N(T)^{\perp}}.$$

This factorization is called the canonical factorization of T.

3. Order automorphisms on the unit interval

$\varphi : \Lambda \longrightarrow \Lambda$: diamond order automorphism. $\varphi =?$

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Let Δ be the set of all semi-linear bijections on $\mathcal{H}$ if the dimension of $\mathcal{H}$ is finite, and let Δ be the set of all bounded invertible linear or conjugate linear operators on $\mathcal{H}$ if the dimension of $\mathcal{H}$ is infinite.

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For $A \in \Delta$, we define $\delta_A(T) = ATA^{-1}$, $\forall T \in \mathcal{B}(\mathcal{H})$. We now define two canonical maps on Λ.

$$\varphi_A^1(T) = P_{\overline{R(\delta_A(T))}} P_{\overline{R(\delta_A(T)^*)}}, \qquad \forall T \in \Lambda;$$

$$\varphi_A^2(T) = P_{\overline{R(\delta_A(T^*))}} P_{\overline{R((\delta_A(T^*))^*)}}, \quad \forall T \in \Lambda.$$

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Let $A \in \Delta$. Then φ_A^1 and φ_A^2 defined as above are automorphisms of the poset $(\Lambda, \leq^{\diamond})$.

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Theorem 3.5

Let $\varphi : \Lambda \longrightarrow \Lambda$ be a map. Then φ is an automorphism of the poset $(\Lambda, \leq^{\diamond})$ if and only if there is some $A \in \Delta$ such that either $\varphi = \varphi_A^1$ or $\varphi = \varphi_A^2$.

Remark

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Let $\varphi : \mathcal{B}(\mathcal{H}) \longrightarrow \mathcal{B}(\mathcal{H})$: star partial order automorphism. If φ is additive, then there are a nonzero constant $\alpha \in \mathbb{C}$ and both unitary operators $U, V \in \mathcal{B}(\mathcal{H})$ or both anti-unitary operators U and V on $\mathcal{H}$ such that $\varphi(A) = \alpha UAV$, $\forall A \in \mathcal{B}(\mathcal{H})$ or $\varphi(A) = \alpha UA^*V$, $\forall A \in \mathcal{B}(\mathcal{H})$.

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In general, unknown.

Definition 4.1

Let $\leq$ be a partial order on a von Neumann algebra $\mathcal{M}$ and let $\mathcal{S} \subseteq \mathcal{M}$ be a subset.

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2. Let C be an upper bound of $\mathcal{S}$. If there are not any upper bound D of $\mathcal{S}$ such that $D \leq C$, then we say that C is a minimal upper bound of $\mathcal{S}$. If for any upper bound D of $\mathcal{S}$, $C \leq D$, then we say that C is the supremum of S and denoted by $\bigvee_{\mathcal{M}} \mathcal{S}$.

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3. If $\mathcal{M} = \mathcal{B}(\mathcal{H})$, then we abbreviate as $\bigvee \mathcal{S}$ if there exists the supremum.

We similarly may define maximal lower bounds and the infimum of $\mathcal{S}$ and denoted by $\bigwedge_{\mathcal{M}} \mathcal{S}(\bigwedge \mathcal{S})$ the infimum of $\mathcal{S}$

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Lemma 4.2

Let $\mathcal{H} = H_1 \oplus H_2$ and $A, B \in \mathcal{B}(\mathcal{H})$ such that

$$A = \begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix}, \quad B = \begin{pmatrix} B_{11} & 0 \\ 0 & B_{22} \end{pmatrix}.$$

Then

1. $A \overset{*}{\wedge} B = \begin{pmatrix} A_{11} \overset{*}{\wedge} B_{11} & 0 \\ 0 & A_{22} \overset{*}{\wedge} B_{22} \end{pmatrix}$.
2. If there is a upper bound for A and B, then $A \overset{*}{\vee} B = \begin{pmatrix} A_{11} \overset{*}{\vee} B_{11} & 0 \\ 0 & A_{22} \overset{*}{\vee} B_{22} \end{pmatrix}$.

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However, there are no supremum and infimum for a bounded subset in $\mathcal{B}(\mathcal{H})$ with respect to the diamond partial order in general.

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Let $\mathcal{S} \subseteq \mathcal{B}(\mathcal{H})$. If $\mathcal{S}$ is bounded with respect to the diamond partial order, then there exists a minimal upper bound for $\mathcal{S}$.

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Let B be an upper bound of $\mathcal{S}$. Put $P = P_{\bigvee\{P_{\overline{R(S)}}:S\in\mathcal{S}\}}$, $Q = P_{\bigvee\{P_{\overline{R(S^*)}}:S\in\mathcal{S}\}}$ and $A = PBQ$. Moreover, put $H_1 = \overline{R(A)}$, $H_2 = P(\mathcal{H}) \ominus H_1$ and $H_3 = \mathcal{H} \ominus P(\mathcal{H})$. $K_1 = P_{\overline{R(A^*)}}$, $K_2 = \mathcal{H} \ominus Q(\mathcal{H})$ and $K_3 = Q(\mathcal{H}) \ominus K_1$. Then

$$\mathcal{H} = K_1 \oplus K_2 \oplus K_3 = H_1 \oplus H_2 \oplus H_3. \qquad (5)$$

Theorem 4.5

Let $T \in \mathcal{B}(K_2, H_2)$ and $S \in \mathcal{B}(K_3, H_3)$ such that both T and S^* with dense ranges. Then

$$B_{T,S} = \begin{pmatrix} A & 0 & 0 \\ 0 & T & 0 \\ 0 & 0 & S \end{pmatrix}$$

with respect to decomposition (5) of $\mathcal{H}$ is an upper bound of $\mathcal{S}$. $B_{T,S}$ is a minimal upper bound if and only if both T and S^* are surjective.

Proposition 4.6

Let $\{A_\alpha\}_{\alpha\in\Lambda}$ be an increasing net in $\mathcal{B}(\mathcal{H})$ and bounded from above with respect to the diamond partial order. Then

1. $A_\alpha \longrightarrow A(SOT)$ for some $A \in \mathcal{B}(\mathcal{H})$ such that $A \leq^\diamond D$ for any upper bound D of $\{A_\alpha\}_{\alpha\in\Lambda}$.
2. A is the supremum of $\{A_\alpha\}_{\alpha\in\Lambda}$ if and only if $P_{A_\alpha} \to P_A(SOT)$ and $Q_\alpha \to Q_A(SOT)$.

Example 4.7

Let $\mathcal{H} = L^2[(0,1]) \oplus L^2([0,1]) \oplus L^2([0,1])$.Put $\Delta_s = [0,s]$, $\Omega_s = (s,1]$ and $f_s(t) = \chi_{\Delta_s}(t) + \frac{1}{2}\chi_{\Omega_s}(t)$, $\forall s \in [0,1)$. We define

$$P_s = \begin{pmatrix} f_s & \frac{1}{2}\chi\Omega_s & 0 \\ \frac{1}{2}\chi\Omega_s & f_s & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; Q_s = \begin{pmatrix} f_s & 0 & \frac{1}{2}\chi\Omega_s \\ 0 & 0 & 0 \\ \frac{1}{2}\chi\Omega_s & 0 & f_s \end{pmatrix}.$$

Then P_s and Q_s are projections and $A_s = P_sQ_s$ is increasing and with an upper bound I. Note that $P = P_{\bigvee\{R(P_sQ_s):0\leq s<1\}} = I \oplus I \oplus 0$ and $Q = P_{\bigvee\{R(Q_sP_s):0\leq s<1\}} = I \oplus 0 \oplus I$. However, $A_s = P_sQ_s \to PQ = A = I \oplus 0 \oplus 0(SOT)$ and A is not an upper bound of $\{A_s : 0 \leq s < 1\}$. By Theorem 2.6, there many minimal upper bounds for $\{A_s : 0 \leq s < 1\}$.

Are there maximal lower bounds for $\mathcal{S}$?

Theorem 4.8

Let $\mathcal{A}$ be a nonempty subset in $\mathcal{B}(\mathcal{H})$. If $B \in \mathcal{B}(\mathcal{H})$ is an upper bound of $\mathcal{A}$ with respect to the diamond partial order, then EBF is a maximal lower bound of $\mathcal{A}$.

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Proposition 4.9

Let $\{A_\alpha\}_{\alpha\in\Lambda}$ be a decreasing net in $\mathcal{B}(\mathcal{H})$ respect to the diamond partial order. Then $A_\alpha \longrightarrow A(SOT)$ and A is a maximal lower bound for $\{A_\alpha\}_{\alpha\in\Lambda}$. A is the infimum of $\{A_\alpha\}_{\alpha\in\Lambda}$ if and only if $P_{A_\alpha} \to P_A(SOT)$ and $Q_\alpha \to Q_A(SOT)$.

Example 4.10

$\mathcal{K}$: separable infinite dimensional Hilbert space.

$C, D \in \mathcal{B}(\mathcal{K})$: injective with dense range.

$M \subset \mathcal{K}$: linear manifold s.t. $R(D^*)\dot{+}M = \mathcal{K}$ and $\overline{M} = \mathcal{K}$.

$\{f_1, f_2, \cdots, f_n, \cdots\} \subset M$: basis for $\mathcal{K}$.

F_n: projection onto $\vee[f_{n+1}, f_{n+2}, \cdots, f_{n+k}, \cdots]$, $\forall n \in \mathbb{N}$.

$F_n \searrow 0(SOT)$.

$\mathcal{H} = \mathcal{K} \oplus \mathcal{K} \oplus \mathcal{K}$, $B = C \oplus D \oplus D^*$.

$P_n = I \oplus I \oplus F_n \searrow P = I \oplus I \oplus 0$.

$Q_n = I \oplus F_n \oplus I \searrow Q = I \oplus 0 \oplus I$.

$A_n = P_n B Q_n \searrow A = C \oplus 0 \oplus 0$.

Put $B_E = EBQ$, for any projection $E \leq P$ such that $EP_A \neq P_A E$.

$B_E \leq^{\diamond} A_n$ for all n.

B_E: not comparable with A.

Thank You!