Crossing Probabilities in 2D Critical Lattice Models

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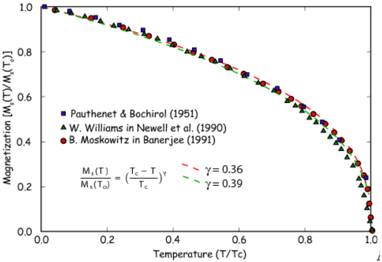
Ising Model

Pierre and Marie Skłodowska-Curie, 1895

Curie temperature [Pierre Curie, 1895]

Ferromagnet exhibits a phase transition by losing its magnetization when heated above a critical temperature.



Ising Model

Ising Model [Lenz 1920]

A model for ferromagnet, to understand the phase transition.

- $G = (V, E)$ a finite graph
- $\sigma \in \{\ominus, \oplus\}^V$
- $H(\sigma) = -\sum_{x \sim y} \sigma_x \sigma_y$

Ising model is the probability measure of inverse temperature $\beta > 0$:

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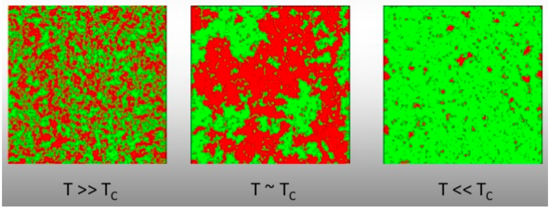

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- $\beta \approx \beta_c$: critical
- $\beta > \beta_c$: chaotic

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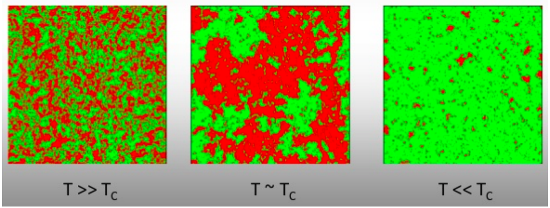

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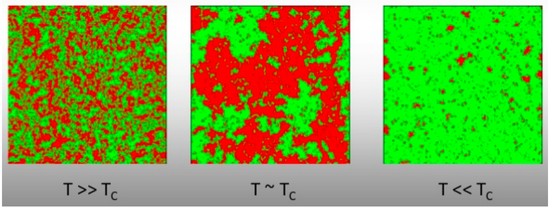

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Ising model on $\mathbb{Z}^2$: $\beta_c = \frac{1}{2}\log(1+\sqrt{2})$.

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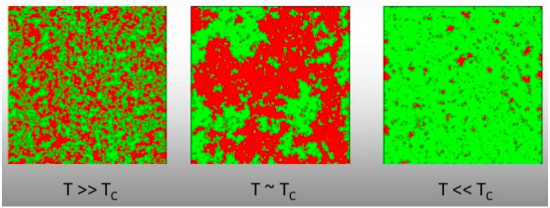

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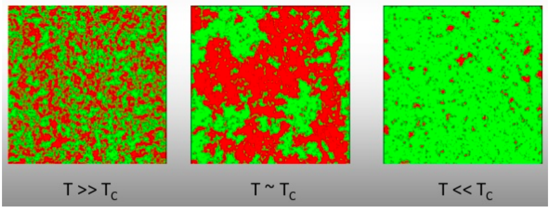

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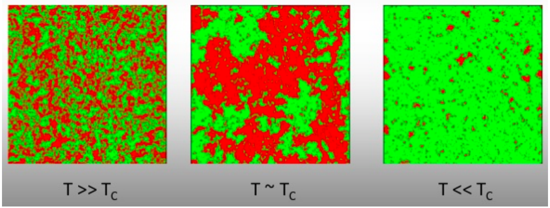

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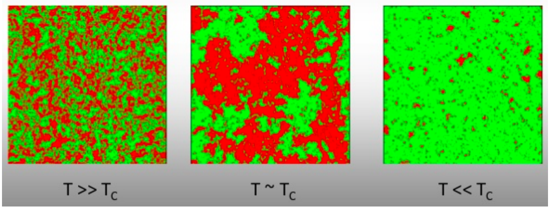

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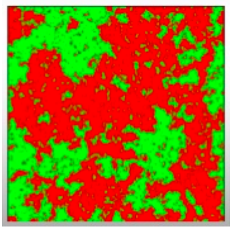
Correlation function

$\mu[\sigma_{z_1}\cdots\sigma_{z_n}] \to \phi(z_1,\ldots,z_n)$.

Schramm Loewner Evolution (SLE)

The law of interfaces is CI.

Conformal Invariance of Interfaces



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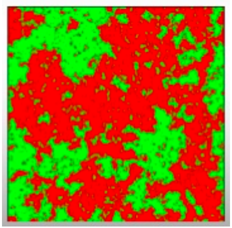

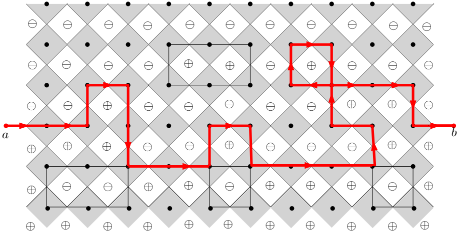

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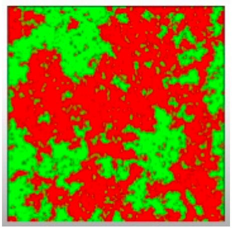

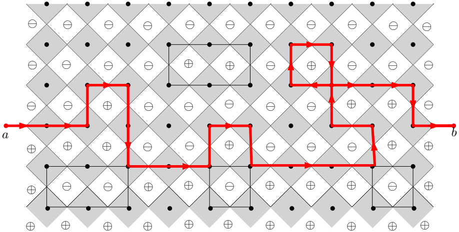

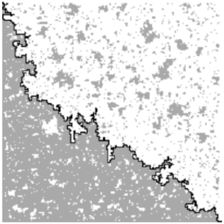

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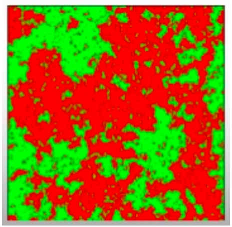
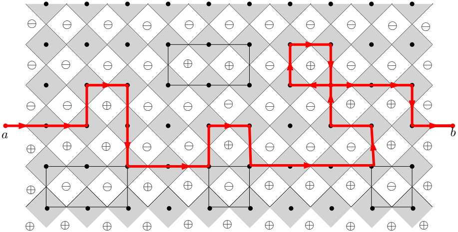
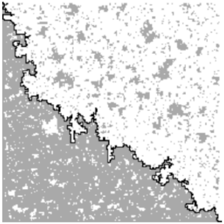




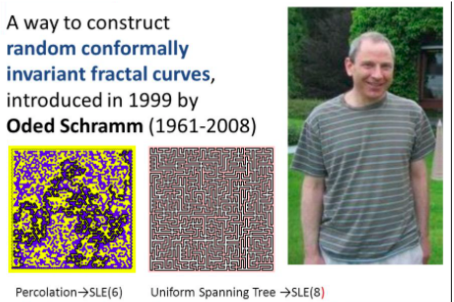


SLE[O. Schramm 1999]

A random fractal curve :

- conformal invariance
- domain Markov property

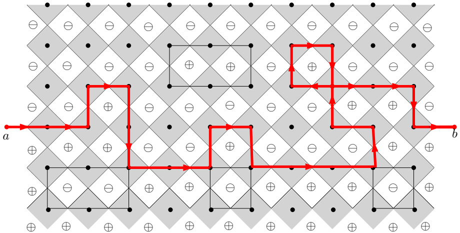
Classification : SLE(κ), $\kappa > 0$.

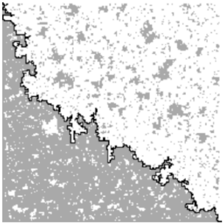


Conformal Invariance in Ising Model

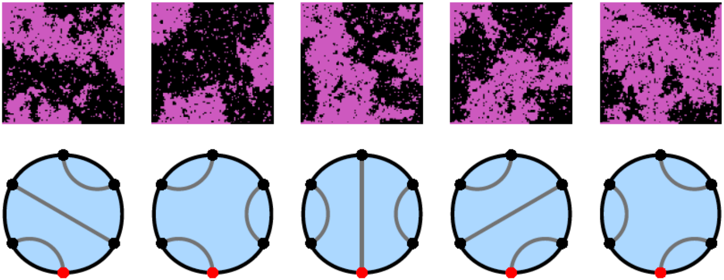
[Chelkak-Smirnov, Inventiones '10]

The interface in critical Ising model on $\mathbb{Z}^2$ with Dobrushin boundary conditions converges weakly to SLE(3).





Crossing Probabilities of Ising Interfaces



Theorem [Peltola-W. '18]

The connection of Ising interfaces forms a planar link pattern $\mathcal{A}_\delta$.

$$\lim_{\delta \to 0} \mathbb{P}[\mathcal{A}_\delta = \alpha] = \frac{\mathcal{Z}_\alpha(\Omega; x_1, \ldots, x_{2N})}{\mathcal{Z}_{Ising}(\Omega; x_1, \ldots, x_{2N})},$$

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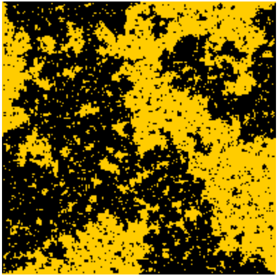
- Conjectured in [Bauer-Bernard-Kytölä, JSP '05].
- Partially solved in [Izyurov, CMP '15].
- Might be related to correlation functions in CFT.

Pure Partition Functions



Pure Partition Functions

$\{\mathcal{Z}_\alpha : \alpha \in \mathrm{LP}\}$ is a collection of smooth functions satisfying PDE, COV, ASY.

PDE : $\left[\frac{\kappa}{2}\partial_i^2 + \sum_{j\neq i}\left(\frac{2}{x_j - x_i}\partial_j - \frac{(6-\kappa)/\kappa}{(x_j-x_i)^2}\right)\right]\mathcal{Z}(x_1, \ldots, x_{2N}) = 0.$

COV : $\mathcal{Z}(x_1, \ldots, x_{2N}) = \prod_{i=1}^{2N}\varphi'(x_i)^h \times \mathcal{Z}(\varphi(x_1), \ldots, \varphi(x_{2N})).$

ASY : $\lim_{x_j, x_{j+1}\to\xi}\frac{\mathcal{Z}_\alpha(x_1,\ldots,x_{2N})}{(x_{j+1}-x_j)^{-2h}} = \mathcal{Z}_{\hat{\alpha}}(x_1, \ldots, x_{j-1}, x_{j+2}, \ldots, x_{2N})$

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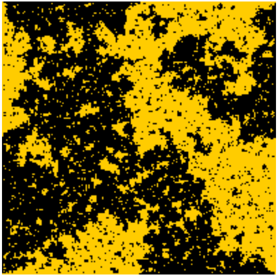

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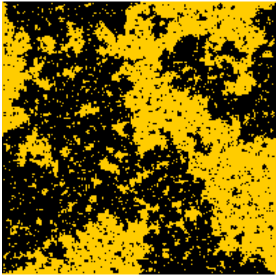
Questions

Existence ? Uniqueness ? Explicit formula ?

Pure Partition Functions



Uniqueness [Flores-Kleban, CMP '15]

If there exist collections of smooth functions satisfying PDE, COV and ASY, they are (essentially) unique.

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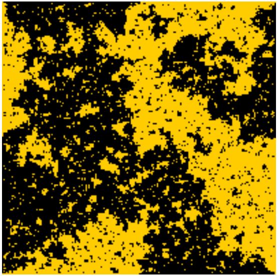

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Existence

- $\kappa \in (0, 8) \setminus \mathbb{Q}$ [Kytöla-Peltola, CMP'16]
- $\kappa \in (0, 4]$ [Peltola-W. CMP'19]
- $\kappa \in (0, 6]$ [W. CMP'20]

- Coulumb gas techniques
- Global multiple SLEs
- Hypergeometric SLE

Pure Partition Functions

Theorem [W. CMP'20]

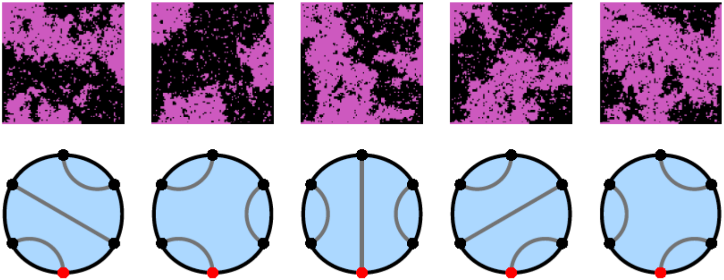
Let $\kappa \in (0, 6]$. There exists a unique collection $\{\mathcal{Z}_\alpha : \alpha \in \mathrm{LP}\}$ of smooth functions satisfying the normalization $\mathcal{Z}_\emptyset = 1$ and

PDE, COV, ASY, POS and PLB

the power law bound : for all $\alpha = \{\{a_1, b_1\}, \ldots, \{a_N, b_N\}\} \in \mathrm{LP}_N$,

$$0 < \mathcal{Z}_\alpha(x_1, \ldots, x_{2N}) \leq \prod_{j=1}^{N} |x_{b_j} - x_{a_j}|^{-2h}.$$

Crossing Probabilities of Ising Interfaces



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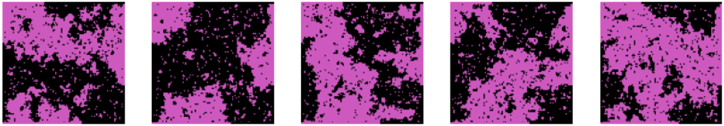
where $\{\mathcal{Z}_\alpha\}$ is the pure partition functions for multiple SLE_3.

Connection Probabilities

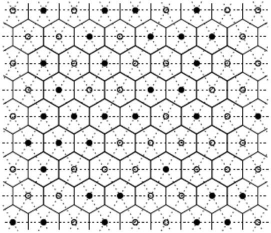


Courtesy to E. Peltola

$$\mathbb{P}[\mathcal{A} = \alpha] = \frac{\mathcal{Z}_\alpha(\Omega; x_1, \ldots, x_{2N})}{\mathcal{Z}^{(N)}(\Omega; x_1, \ldots, x_{2N})}, \qquad \mathcal{Z}^{(N)} = \sum_{\alpha \in \mathrm{LP}_N} \mathcal{Z}_\alpha.$$

- LERWs in UST : $\kappa = 2$. [Karrila-Kytölä-Peltola, CMP'19]
- Multiple Ising interfaces : $\kappa = 3$. [Peltola-W. '18]
- Multiple level lines of GFF : $\kappa = 4$. [Peltola-W. CMP'19]
- Multiple percolation interfaces : $\kappa = 6$. [Peltola-W. '20+]

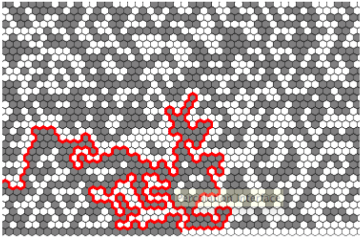
Percolation



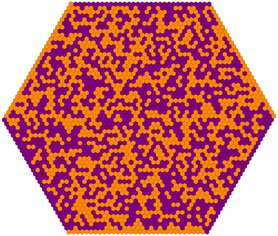
Site percolation on triangular lattice : each site is chosen independently to be black or white with equal probability $1/2$.

Thm. [Smirnov, '01]

The interface of critical site percolation on triangular lattice converges weakly to SLE(6).



Crossing Probabilities in Critical Percolation



- When $N = 2$: Cardy's formula [Smirnov '01]

Theorem [Peltola-W. '20+]

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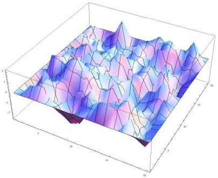
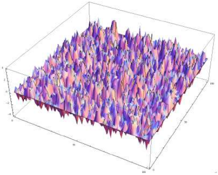
where $\{\mathcal{Z}_\alpha : \alpha \in \mathrm{LP}_N\}$ is the pure partition functions for multiple SLE_6.

DGFF (Discrete Gaussian Free Field)





DGFF with mean zero : a measure h on functions $\rho : D \to \mathbb{R}$ and $\rho = 0$ on ∂D with density

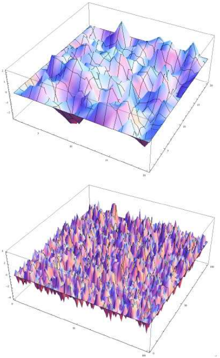
$$\frac{1}{\mathcal{Z}} \exp(-\frac{1}{2} \sum_{x \sim y} (\rho(x) - \rho(y))^2).$$

- For each vertex x, $h(x)$ Gaussian r.v.
- Covariance : Green's function for SRW
- Mean value : zero.

DGFF with mean h_∂ : DGFF with mean zero plus a harmonic function h_∂.

- For each vertex x, $h(x)$ Gaussian r.v.
- Covariance : Green's function for SRW
- Mean value : $h_\partial(x)$

GFF (Continuum Gaussian Free Field)



DGFF $\to$ GFF h

- (h, ρ) Gaussian r.v.
- Covariance :

$$cov((h,\rho_1),(h,\rho_2)) = \iint dxdy G_D(x,y)\rho_1(x)\rho_2(y).$$

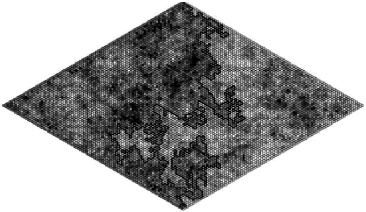
- Mean value : $\mathbb{E}((h,\rho)) = (h_\partial, \rho)$.

GFF is :

- height functions of dimer model, domino tiling...
- starting point for quantum field theory...

Conformal Invariance
Domain Markov Property

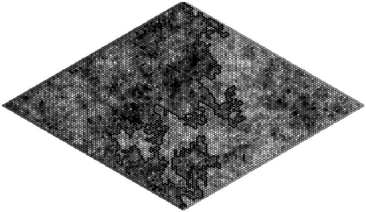
Level lines of DGFF



[Schramm-Sheffield, ACTA'09]

- DGFF with boundary value $+\lambda$ on $\mathbb{R}_+$ and $-\lambda$ on $\mathbb{R}_-$
- γ^δ : the level line of DGFF with height zero
- γ^δ converges in distribution to SLE_4 as δ goes to zero

Level lines of DGFF



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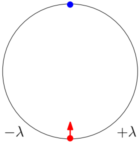
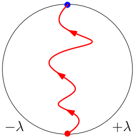
- $\rightarrow$ SLE_4 is the "level line" of GFF with height zero

Interacting level lines





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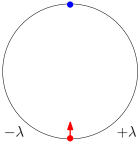
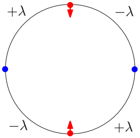
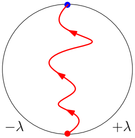
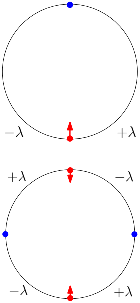
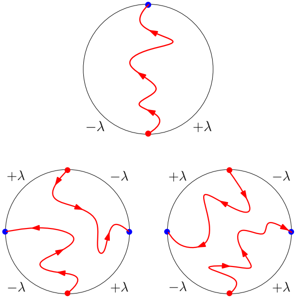




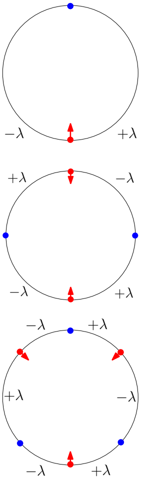


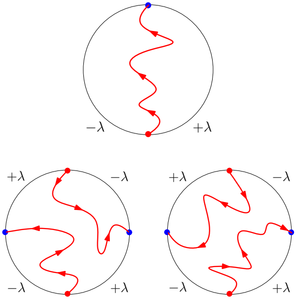
Interacting level lines



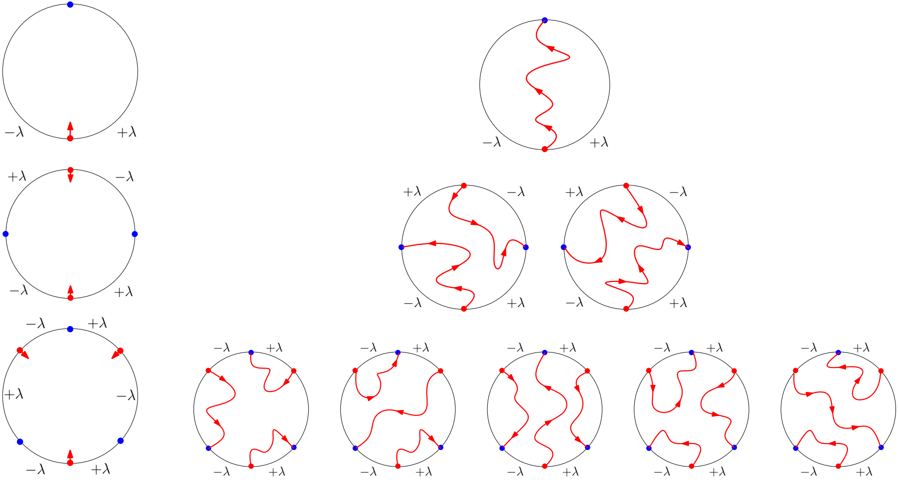


Interacting level lines

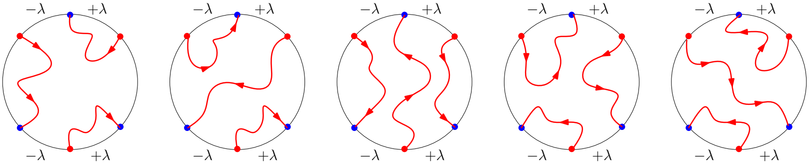




Interacting level lines



Connection Probabilities



- $2N$ marked points
- N level lines
- LP_N : planar link patterns

Theorem [Peltola-W. CMP'19]

The connection of level lines of GFF forms a planar link pattern $\mathcal{A}$: for any $\alpha \in \mathrm{LP}_N$,

$$\mathbb{P}[\mathcal{A} = \alpha] = \frac{\mathcal{Z}_\alpha(\Omega; x_1, \ldots, x_{2N})}{\mathcal{Z}_{\mathrm{GFF}}(\Omega; x_1, \ldots, x_{2N})}, \quad \mathcal{Z}_{\mathrm{GFF}} = \sum_{\alpha \in \mathrm{LP}_N} \mathcal{Z}_\alpha,$$

where $\{\mathcal{Z}_\alpha : \alpha \in \mathrm{LP}_N\}$ is the pure partition functions for multiple SLE_4.

Connection Probabilities

Theorem [Kenyon-Wilson, '11]

The collection $\{\mathcal{Z}_\alpha\}$ are explicit when $\kappa = 4$:

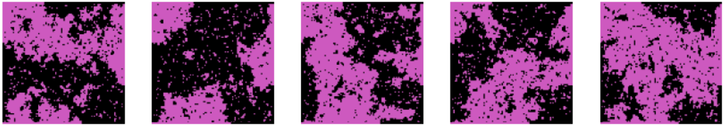
$$\mathcal{Z}_\alpha(x_1, \ldots, x_{2N}) = \sum_{\beta \in \mathrm{LP}_N} \mathcal{M}^{-1}_{\alpha,\beta} \mathcal{U}_\beta(x_1, \ldots, x_{2N}),$$

where

$$\mathcal{U}_\beta(x_1, \ldots, x_{2N}) := \prod_{1 \le i < j \le 2N} (x_j - x_i)^{\frac{1}{2}\vartheta_\beta(i,j)},$$

$$\mathcal{M}_{\alpha,\beta} = 1\{\alpha \leftarrow \beta\}, \qquad \mathcal{M}^{-1}_{\alpha,\beta} = (-1)^{|\alpha/\beta|} \#\mathcal{C}(\alpha/\beta) 1\{\alpha \preceq \beta\}.$$

Connection Probabilities



Courtesy to E. Peltola

$$\mathbb{P}[\mathcal{A}=\alpha]=\frac{\mathcal{Z}_\alpha(\Omega;x_1,\ldots,x_{2N})}{\mathcal{Z}^{(N)}(\Omega;x_1,\ldots,x_{2N})},\qquad \mathcal{Z}^{(N)}=\sum_{\alpha\in\mathrm{LP}_N}\mathcal{Z}_\alpha.$$

- LERWs in UST : $\kappa=2$. [Karrila-Kytölä-Peltola, CMP'19]
- Multiple Ising interfaces : $\kappa=3$. [Peltola-W. '18]
- Multiple level lines of GFF : $\kappa=4$. [Peltola-W. CMP'19]
- Multiple percolation interfaces : $\kappa=6$. [Peltola-W. '20+]

References Thanks !

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