Smoothness functions spaces on metric spaces

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Outline

1. How to measure smoothness
2. Function spaces on metric spaces with smoothness order ≤ 1
3. Characterize Sobolev spaces via ball averages
4. Characterize Besov-Triebel-Lizorkin spaces via ball averages

1. How to measure smoothness

How to measure smoothness

• Function spaces with smoothness, such as Zygmund spaces, Lipschitz spaces, Sobolev spaces, Besov spaces and Triebel-Lizorkin spaces, have found wide applications in various areas of mathematics.

• Basic problem: How to measure smoothness?

• Classical tools on $\mathbb{R}^n$: derivatives, differences, Fourier transform, interpolation, decompositions (atoms, molecules, wavelets...), . . .

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Zygmund spaces: differences

Zygmund spaces $C^m(\mathbb{R}^n)$, $m \in \mathbb{N}$

$f \in C^m(\mathbb{R}^n) \Longleftrightarrow f \in C^{m-1}(\mathbb{R}^n)$ with

$$\|f\|_{C^m(\mathbb{R}^n)} := \|f\|_{C^{m-1}(\mathbb{R}^n)} + \sum_{|\gamma|=m-1} \sup_{h\in\mathbb{R}^n\setminus\{0\}} \frac{\|\Delta_h^2 \partial^\gamma f\|_{C^0(\mathbb{R}^n)}}{|h|} < \infty.$$

- $\Delta_h^1 g(x) := g(x+h) - g(x)$
  $\Delta_h^2 g(x) := \Delta_h^1 \Delta_h^1 g(x) = g(x+2h) + g(x) - 2g(x+h)$
- $C^0(\mathbb{R}^n)$ – bounded and uniformly continuous functions

Sobolev spaces: derivatives

Sobolev spaces $\dot{W}^{m,p}(\mathbb{R}^n)$ & $W^{m,p}(\mathbb{R}^n)$, $m \in \mathbb{N}$, $p \in (1, \infty)$

$f \in \dot{W}^{m,p}(\mathbb{R}^n) \Longleftrightarrow f$ is locally integrable and $\partial^\gamma f \in L^p(\mathbb{R}^n)$ for all $|\gamma| = m$, with

$$\|f\|_{\dot{W}^{m,p}(\mathbb{R}^n)} := \sum_{|\gamma|=m} \|\partial^\gamma f\|_{L^p(\mathbb{R}^n)}.$$

$f \in W^{m,p}(\mathbb{R}^n) \Longleftrightarrow f \in L^p(\mathbb{R}^n)$ and $\partial^\gamma f \in L^p(\mathbb{R}^n)$ for all $|\gamma| \le m$, with

$$\|f\|_{W^{m,p}(\mathbb{R}^n)} := \sum_{|\gamma|\le m} \|\partial^\gamma f\|_{L^p(\mathbb{R}^n)}.$$

$* \ W^{m,p}(\mathbb{R}^n) = \dot{W}^{m,p}(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$.

Fractional Sobolev spaces: Fourier analytic tool

- Recall that $(\partial^\gamma f)^\wedge(\xi) = \xi^\gamma \widehat{f}(\xi)$

Fractional Sobolev spaces $\dot{W}^{\alpha,p}(\mathbb{R}^n)$ & $W^{\alpha,p}(\mathbb{R}^n)$, $\alpha \in (0,\infty)$, $p \in (1,\infty)$

$f \in \dot{W}^{\alpha,p}(\mathbb{R}^n) \iff f$ is locally integrable and $I_\alpha f \in L^p(\mathbb{R}^n)$, with

$$\|f\|_{\dot{W}^{\alpha,p}(\mathbb{R}^n)} := \|I_\alpha f\|_{L^p(\mathbb{R}^n)}.$$

Moreover,

$$W^{\alpha,p}(\mathbb{R}^n) = L^p(\mathbb{R}^n) \cap \dot{W}^{\alpha,p}(\mathbb{R}^n).$$

- $(I_\alpha f)^\wedge(\xi) := |\xi|^\alpha \widehat{f}(\xi), \ \forall\, \xi \in \mathbb{R}^n \setminus \{0\}$

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Besov spaces: Fourier analytic tool

- $\mathcal{S}(\mathbb{R}^n)$ — Schwartz functions

$\mathcal{S}_\infty(\mathbb{R}^n)$ — the set of $f \in \mathcal{S}(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} f(x)x^\gamma\, dx = 0$ for all γ

- Let $\{\varphi_j\}_{j\in\mathbb{Z}} := \{2^{jn}\varphi(2^j\cdot)\}$ with $\varphi \in \mathcal{S}(\mathbb{R}^n)$ satisfying

$$\operatorname{supp}\widehat{\varphi} \subset B(0,2) \setminus B(0,2^{-1}), \quad |\widehat{\varphi}(\xi)| \ge C > 0 \text{ if } \frac{3}{5} \le |\xi| \le \frac{5}{3}.$$

(Homogeneous) Besov spaces $\dot{B}^\alpha_{p,q}(\mathbb{R}^n)$, $\alpha \in \mathbb{R}$, $p, q \in (0,\infty]$

$f \in \dot{B}^\alpha_{p,q}(\mathbb{R}^n) \Longleftrightarrow f \in \mathcal{S}'_\infty(\mathbb{R}^n)$ so that

$$\|f\|_{\dot{B}^\alpha_{p,q}(\mathbb{R}^n)} := \left\{\sum_{j\in\mathbb{Z}} 2^{j\alpha q}\|\varphi_j * f\|^q_{L^p(\mathbb{R}^n)}\right\}^{1/q} < \infty.$$

$* \; 2^{j\alpha}(\varphi_j * f)^\wedge(\xi) = 2^{j\alpha}\widehat{\varphi}(2^{-j}\xi)\widehat{f}(\xi)$

Triebel-Lizorkin spaces: Fourier analytic tool

(Homogeneous) Triebel-Lizorkin $\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)$, $\alpha \in \mathbb{R}$, $p \in (0, \infty)$, $q \in (0, \infty]$

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$$\|f\|_{\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)} := \left\| \left\{ \sum_{j\in\mathbb{Z}} 2^{j\alpha q} |\varphi_j * f|^q \right\}^{1/q} \right\|_{L^p(\mathbb{R}^n)} < \infty.$$

- If $p > 1$, then $A^{\alpha}_{p,q}(\mathbb{R}^n) = L^p(\mathbb{R}^n) \cap \dot{A}^{\alpha}_{p,q}(\mathbb{R}^n)$ with $A \in \{B, F\}$.
- The tools used to define the above function spaces rely on the **linear and differential structures** of $\mathbb{R}^n$.

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Metric measure spaces

Metric measure space

Let X be a non-empty set, d a metric on X, namely, a non-negative function on $X \times X$ satisfying: for any $x,\ y,\ z \in X$,

1. $d(x,y) = 0 \Longleftrightarrow x = y$;
2. $d(x,y) = d(y,x)$;
3. $d(x,z) \le d(x,y) + d(y,z)$.

Assume that μ is a Borel measure on X. Then (X,d,μ) is called a metric measure space.

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• In general, a metric measure space has **no** classical linear and differential structure as $\mathbb{R}^n$

• Due to the lackness of many classical important tools (addition, high order difference, derivatives, Fourier transforms, ...), the main difficulty for developing function spaces with smoothness on metric measure spaces is to find suitable tools to describe regularity/smoothness.

• From 90's, functions spaces with smoothness order ≤ 1 on metric measure spaces have received great progresses, e. g, Sobolev spaces via Hajłasz gardients or upper gradients

2. Function spaces on metric spaces with smoothness order ≤ 1

2.1 Hajłasz-Sobolev spaces

Bojarski 1991

If $p \in (1, \infty)$ and $f \in \dot{W}^{1,p}(\mathbb{R}^n)$, then for all $x, y \in \mathbb{R}^n$,

$$|f(x) - f(y)| \lesssim |x - y| \left[M_{|x-y|}(\nabla f)(x) + M_{|x-y|}(\nabla f)(y)\right]$$

- For $R \in (0, \infty]$, $g \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$,

$$M_R(g)(x) := \sup_{r<R} \frac{1}{r^n} \int_{B(x,r)} |g(y)|\, dy, \quad x \in \mathbb{R}^n.$$

- Note that $M_R(\nabla f) \in L^p(\mathbb{R}^n)$

[B91] **B. Bojarski**, *Remarks on some geometric properties of Sobolev mappings*, Functional analysis & related topics (Sapporo, 1990), 65-76, World Sci. Publ., River Edge, NJ, 1991.

Pointwise characterization of $\dot{W}^{1,p}(\mathbb{R}^n)$

Hajłasz 1996

Let $p \in (1, \infty)$ and f be a measurable function. Then $f \in \dot{W}^{1,p}(\mathbb{R}^n)$ $\Longleftrightarrow$ there exists a $0 \le g \in L^p(\mathbb{R}^n)$ such that for a. e. $x,\ y \in \mathbb{R}^n$,

$$|f(x) - f(y)| \lesssim |x - y|[g(x) + g(y)].$$

Moreover, $|\nabla f| \lesssim g$ *a. e.*

- Note that the above pointwise inequality can be easily extended to a general metric space

[H96] **P. Hajłasz**, *Sobolev spaces on an arbitrary metric space*, Potential Anal. 5 (1996), 403-415.

Hajłasz gradients [H96]

Let (X, d, μ) be a metric measure space, f is a measurable function on X. A non-negative function g is called a Hajłasz gradient of f, if it satisfies

$$|f(x) - f(y)| \lesssim d(x, y)[g(x) + g(y)].$$

for a. e. x, $y \in X$.

Hajłasz-Sobolev spaces [H96]

Let (X, d, μ) be a metric measure space and $p \in (1, \infty)$. The Hajłasz-Sobolev space $\dot{M}^{1,p}(X)$ is the set of all measurable functions f on X which have Hajłasz gradient $g \in L^p(\mathbb{R}^n)$. Moreover, $\|f\|_{\dot{M}^{1,p}(X)} := \inf_g \|g\|_{L^p(X)}$. The inhomogeneous counterpart $M^{1,p}(X)$ is defined as $L^p(X) \cap \dot{M}^{1,p}(X)$.

- [H96] $\dot{M}^{1,p}(\mathbb{R}^n) = \dot{W}^{1,p}(\mathbb{R}^n), \quad M^{1,p}(\mathbb{R}^n) = W^{1,p}(\mathbb{R}^n)$

Fractional Hajłasz gradients & Sobolev spaces [Hu03,Y03]

- For $\alpha \in (0,1]$, a α-Hajłasz gradient of f is a non-negative function g satisfying

$$|f(x) - f(y)| \lesssim [d(x,y)]^{\alpha}[g(x) + g(y)] \quad \text{a. e. } x,\, y \in X.$$

- The fractional Hajłasz-Sobolev space $\dot{M}^{\alpha,p}(X)$ with $p \in [1,\infty)$ is

$$\{f \text{ measurable} : \ f \text{ has } \alpha\text{-Hajłasz gradient } g \in L^p(\mathbb{R}^n)\}$$

  with $\|f\|_{\dot{M}^{\alpha,p}(X)} := \inf_g \|g\|_{L^p(X)}$.
- $M^{\alpha,p}(X) := L^p(X) \cap \dot{M}^{\alpha,p}(X)$.

[Hu03] **J. Hu**, *A note on Hajłasz-Sobolev spaces on fractals*, J. Math. Anal. Appl. 280 (2003), 91-101.

[Y03] **D. Yang**, *New characterizations of Hajłasz-Sobolev spaces on metric spaces*, Sci. China Ser. A 46 (2003), 675-689.

- Recall that $\dot{M}^{1,p}(\mathbb{R}^n) = \dot{W}^{1,p}(\mathbb{R}^n) = \dot{F}^1_{p,2}(\mathbb{R}^n)$

- [Y03] Surprisingly, for $\alpha \in (0,1)$,

$$\dot{M}^{\alpha,p}(\mathbb{R}^n) = \dot{F}^{\alpha}_{p,\infty}(\mathbb{R}^n) \supsetneq \dot{F}^{\alpha}_{p,2}(\mathbb{R}^n) = \dot{W}^{\alpha,p}(\mathbb{R}^n)$$

Disadvantage of Hajłasz gradients

- $f = c$ a.e. on a measurable set $F \Longrightarrow \nabla f = 0$ a.e. on F
- Not true for Hajłasz gradients in general

  Recall

  $$|f(x) - f(y)| \lesssim |x - y| \left[M_{|x-y|}(\nabla f)(x) + M_{|x-y|}(\nabla f)(y) \right]$$

  for $f \in \dot{W}^{1,p}(\mathbb{R}^n)$.

2.2 Newton Sobolev spaces

- A curve $\gamma : [a, b] \to X$ is a continuous mapping from an interval $[a, b]$ into X. If $\ell(\gamma) < \infty$, then γ is rectifiable

p-Modulus

For a collection Γ of curves in X, its p-Modulus with $p \in [1, \infty)$ is given by

$$\mathrm{Mod}_p(\Gamma) := \inf_{\rho \in F(\Gamma)} \|\rho\|^p_{L^p(X)},$$

where

$$F(\Gamma) := \Big\{ \rho \text{ nonnegative Borel measurable on } X : \int_\gamma \rho\, ds \geq 1,\ \forall \gamma \in \Gamma \Big\}$$

Upper gradients [HK98,KM98]

Let u be a measurable function on X. A Borel measurable non-negative function g on $\mathcal{X}$ is an upper gradient of u if

$$(*) \qquad |u(\gamma(b)) - u(\gamma(a))| \le \int_\gamma g\, ds$$

holds for all non-constant compact rectifiable curves $\gamma : [a, b] \to X$. Moreover, if $(*)$ fails only on a family Γ with $\mathrm{Mod}_p(\Gamma) = 0$, then g is called a *p-weak upper gradient of u*.

[HK98] **J. Heinonen and P. Koskela**, *Quasiconformal maps in metric spaces with controlled geometry*, Acta Math. 181 (1998), 1-61.

[KM98] **P. Koskela and P. MacManus**, *Quasiconformal mappings and Sobolev spaces*, Studia Math. 131 (1998), 1-17.

Newton-Sobolev spaces [S00]

Let $p \in [1, \infty)$. Define $\widetilde{N}^{1,p}(X)$ as the set of all $u \in L^p(X)$ which has a p-weak upper gradient $g \in L^p(X)$, with

$$\|u\|_{\widetilde{N}^{1,p}(X)} := \|u\|_{L^p(X)} + \inf_g \|g\|_{L^p(X)},$$

where the infimum is taken over all p-weak upper gradients g of u.

Define $u \sim v \iff \|u - v\|_{\widetilde{N}^{1,p}(X)} = 0$, and the Newton-Sobolev space as

$$N^{1,p}(X) := \widetilde{N}^{1,p}(X)/\sim, \quad \|u\|_{N^{1,p}(X)} := \|u\|_{\widetilde{N}^{1,p}(X)}.$$

[S00] **N. Shanmugalingam**, *Newtonian spaces: an extension of Sobolev spaces to metric measure spaces*, Rev. Mat. Iberoam. 16 (2000), 243-279.

- Locality: $f \equiv c$ *a. e.* $\Longrightarrow$ weak upper gradient $g = 0$ *a. e.*

Doubling measure

The measure μ is doubling, if $\mu(B(x,2r)) \leq C\mu(B(x,r))$.

Poincaré inequality

A space X is said to support a weak $(1,p)$-Poincaré inequality, if $\exists$ $C > 0$ s.t. $\forall$ open balls B in X, $u \in L^1(B)$ and upper gradient g of u,

$$\frac{1}{\mu(B)} \int_B |u - u_B| \, d\mu \leq C \, \mathrm{diam}(B) \left(\frac{1}{\mu(\tau B)} \int_B g^p \, d\mu \right)^{1/p}$$

for some $\tau \geq 1$, where $u_B := \frac{1}{\mu(B)} \int_B u \, d\mu$.

$*$ $\tau = 1$: $(1,p)$-Poincaré inequality

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$*$ $\tau = 1$: $(1,p)$-Poincaré inequality

Relation: Hajłasz and Newton Sobolev spaces

Theorem [S00]

Let $p \in [1, \infty)$. If μ is doubling, X is complete and supports a weak $(1, p)$-Poincaré inequality, then

$$N^{1,p}(X) = M^{1,p}(X).$$

In particular,

$$N^{1,p}(\mathbb{R}^n) = M^{1,p}(\mathbb{R}^n) = W^{1,p}(\mathbb{R}^n).$$

2.3 Besov-Triebel-Lizorkin spaces via ATI

• Recall that Besov and Triebel-Lizorkin spaces on $\mathbb{R}^n$ are defined via $\mathcal{S}_\infty(\mathbb{R}^n)$, $\mathcal{S}'_\infty(\mathbb{R}^n)$ and

$$\varphi_j * f(x) := \int_{\mathbb{R}^n} \varphi_j(x-y) f(y)\, dy.$$

The set $\{\varphi_j\}_j$ can be related to an approximation to identity.

$\dot{B}^\alpha_{p,q}(\mathbb{R}^n)$, $\alpha \in \mathbb{R}$, $p, q \in (0, \infty]$

$f \in \dot{B}^\alpha_{p,q}(\mathbb{R}^n) \iff f \in \mathcal{S}'_\infty(\mathbb{R}^n)$ so that

$$\|f\|_{\dot{B}^\alpha_{p,q}(\mathbb{R}^n)} := \left\{ \sum_{j\in\mathbb{Z}} 2^{j\alpha q} \|\varphi_j * f\|^q_{L^p(\mathbb{R}^n)} \right\}^{1/q} < \infty.$$

Test functions, distributions and ATI

• In 2008, Han, Müller and Yang [HMY08] constructed test functions, distributions and approximations to identity (ATI) on RD spaces, i. e., metric spaces whose measure is doubling and also reverse doubling ($\exists\ \widetilde{C} \in (0, 1]$ and $\kappa > 0$ so that

$$\widetilde{C}\lambda^{\kappa}\mu(B(x, r)) \le \mu(B(x, \lambda r)), \quad \forall \lambda > 1.)$$

[HMY08]**Y. Han, D. Müller and D. Yang**, *A theory of Besov and Triebel–Lizorkin spaces on metric measure spaces modeled on Carnot–Carathéodory spaces*, Abstr. Appl. Anal. 2008, Art. ID 893409, 1–250.

Test functions

Let $x_0 \in \mathcal{X}$, $r \in (0, \infty)$, $\beta \in (0, 1]$ and $\gamma \in (0, \infty)$. A function φ on X is called a test function of type (x_0, r, β, γ) if there exists a positive constant C such that

(i) $|\varphi(x)| \le C \frac{1}{V_r(x_0)+V_r(x)+V(x_0,x)} [\frac{r}{r+d(x_0,x)}]^\gamma$ for any $x \in \mathcal{X}$;

(ii) $|\varphi(x) - \varphi(y)| \le C[\frac{d(x,y)}{r+d(x_0,x)}]^\beta \frac{1}{V_r(x_0)+V_r(x)+V(x_0,x)} [\frac{r}{r+d(x_0,x)}]^\gamma$ for any $x, y \in \mathcal{X}$ satisfying that $d(x, y) \le [r + d(x_0, x)]/2$.

These functions are denoted by $\mathcal{G}(x_0, r, \beta, \gamma)$. Let $\mathring{\mathcal{G}}(x_0, r, \beta, \gamma)$ be its subspace of φ satisfying $\int_X \varphi d\mu = 0$.

- $V_r(x) := \mu(B(x, r))$, $V(x, y) := \mu(B(x, d(x, y)))$
- $\mathcal{G}(x_0, r, \beta, \gamma) = \mathcal{G}(\widetilde{x_0}, \widetilde{r}, \beta, \gamma)$, and so we write $\mathcal{G}(\beta, \gamma) := \mathcal{G}(x_0, r, \beta, \gamma)$
- For $\epsilon \in (0, 1]$, $\beta, \gamma \in (0, \epsilon]$, let $\mathcal{G}_0^\epsilon(\beta, \gamma)$ be the closure of $\mathcal{G}(\epsilon, \epsilon)$ in $\mathcal{G}(\beta, \gamma)$, and define $\mathring{\mathcal{G}}_0^\epsilon(\beta, \gamma)$ similarly.

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(ii) $|\varphi(x) - \varphi(y)| \le C[\frac{d(x,y)}{r+d(x_0,x)}]^\beta \frac{1}{V_r(x_0)+V_r(x)+V(x_0,x)} [\frac{r}{r+d(x_0,x)}]^\gamma$ for any $x, y \in \mathcal{X}$ satisfying that $d(x, y) \le [r + d(x_0, x)]/2$.

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Approximations to identity (ATI)

Let $\epsilon \in (0, 1]$. A sequence $\{S_k\}_{k\in\mathbb{Z}}$ of bounded linear integral operators on $L^2(X)$ is an approximation of the identity (ATI) of order ϵ, if the kernel $S_k(x, y)$ satisfies

(i) $S_k(x, y) = 0$ if $d(x, y) > C_1 2^{-k}$ and $|S_k(x, y)| \leq C_2 \frac{1}{V_{2^{-k}}(x)+V_{2^{-k}}(x)}$;

(ii) if $d(x, x') \leq \max\{C_1, 1\}2^{1-k}$ then

$$|S_k(x, y) - S_k(x', y)| \leq C_2 2^{k\epsilon_1}[d(x, x')]^{\epsilon_1} \frac{1}{V_{2^{-k}}(x) + V_{2^{-k}}(x)};$$

(iii) Property (ii) holds with x and y interchanged;

(iv) if $d(x, x') \leq \max\{C_1, 1\}2^{1-k}$ and $d(x, x') \leq \max\{C_1, 1\}2^{1-k}$, then

$$|[S_k(x, y) - S_k(x, y')] - [S_k(x', y) - S_k(x', y')]| \leq C_2 2^{2k\epsilon} \frac{[d(x, x')]^{\epsilon}[d(y, y')]^{\epsilon}}{V_{2^{-k}}(x) + V_{2^{-k}}(x)}$$

(v) $\int_X S_k(x, z)\, d\mu(z) = 1 = \int_X S_k(z, y)\, d\mu(z)$.

• Let $D_k := S_k - S_{k-1}$ and $D_k(f) := \int_X D_k(\cdot, y)f(y)\,d\mu(y)$

Besov-Triebel-Lizorkin spaces via ATI [HMY08]

Let $\epsilon \in (0, 1)$, $|\alpha| < \epsilon$ and $p, q \in (\max\{n/(n+\epsilon), n/(n+\epsilon+\alpha)\}, \infty]$. The Besov space $\dot{B}^{\alpha}_{p,q}(X)$ and Triebel-Lizorkin space $\dot{F}^{\alpha}_{p,q}(X)$ consist, respectively, of all $f \in (\mathring{\mathcal{G}}^{\epsilon}_0(\beta,\gamma))'$ for some $\beta, \gamma \in (0, \epsilon)$ such that

$$\|f\|_{\dot{B}^{\alpha}_{p,q}(X)} := \left\{\sum_{k\in\mathbb{Z}} 2^{k\alpha q}\|D_k(f)\|^q_{L^p(X)}\right\}^{1/q} < \infty$$

and

$$\|f\|_{\dot{F}^{\alpha}_{p,q}(X)} := \left\|\left\{\sum_{k\in\mathbb{Z}} 2^{k\alpha q}|D_k(f)|^q\right\}^{1/q}\right\|_{L^p(X)} < \infty.$$

Besov-Triebel-Lizorkin spaces via exp-ATI

• Recently, via a new class of approximation to identity with exponential decay constructed in [HLYY], Wang et el. [WHHY] introduced Besov and Triebel-Lizorkin spaces on spaces of homogeneous type (The reverse doubling condition is not needed)

[HLYY] **Z. He, L. Liu, D. Yang and W. Yuan**, *New Calderón reproducing formulae with exponential decay on spaces of homogeneous type*, Sci. China Math. 62 (2019), 283–350.

[WHHY] **F. Wang, Y. Han, Z. He and D. Yang**, *Besov spaces and Triebel–Lizorkin spaces on spaces of homogeneous type with their applications to boundedness of Calderón–Zygmund operators*, Submitted.

2.4 Besov-Triebel-Lizorkin via Hajłasz gradients

- [KYZ11] Hajłasz-Besov and Hajłasz-Triebel-Lizorkin spaces with smoothness $\alpha \in (0, 1]$ on RD-spaces

  $*$ Tool: Hajłasz gradient sequence, i. e., $\{g_k\}_{k\in\mathbb{Z}}$ satisfies that

$$|f(x) - f(y)| \leq [d(x, y)]^\alpha [g_k(x) + g_k(y)], \quad a.\,e.\; d(x, y) \sim 2^{-k}$$

[KYZ11] **P. Koskela, D. Yang and Y. Zhou**, *Pointwise characterizations of Besov and Triebel-Lizorkin spaces and quasiconformal mappings*, Adv. Math. 226 (2011), 3579-3621.

[KYZ11] Hajłasz Besov-Triebel-Lizorkin spaces

Let $\alpha \in (0, 1]$, $p \in (0, \infty)$ and $q \in (0, \infty]$. The Hajłasz Besov space $\dot{N}^{\alpha}_{p,q}(X)$ and Hajłasz Triebel-Lizorkin space $\dot{M}^{\alpha}_{p,q}(X)$ are defined, respectively, as the set of all locally integrable functions f which have Hajłasz gradient sequence $\{g_k\}_{k\in\mathbb{Z}}$ in $\ell^q(L^p)$ and $L^p(\ell^q)$, that is

$$\|f\|_{\dot{N}^{\alpha}_{p,q}(X)} := \inf_{\{g_k\}_{k\in\mathbb{Z}}} \left\{ \sum_{k\in\mathbb{Z}} \|g_k\|^q_{L^p(X)} \right\}^{1/q}$$

and

$$\|f\|_{\dot{M}^{\alpha}_{p,q}(X)} := \inf_{\{g_k\}_{k\in\mathbb{Z}}} \left\| \left\{ \sum_{k\in\mathbb{Z}} |g_k|^q \right\}^{1/q} \right\|_{L^p(X)}$$

are finite.

- Much more simple definitions than B and F spaces defined via ATI
- Coincidence with B and F spaces defined via ATI
- Application: Invariance of quasi-conformal mappings

Question

How to define function spaces with smoothness order > 1 on metric measure space?

- Much more simple definitions than B and F spaces defined via ATI
- Coincidence with B and F spaces defined via ATI
- Application: Invariance of quasi-conformal mappings

Question

How to define function spaces with smoothness order > 1 on metric measure space?

3. Characterize Sobolev spaces via ball averages

A classical characterization of Sobolev spaces

Theorem

Let $\alpha \in (0, 1)$ and $p \in (1, \infty)$. Then the fractional Sobolev space $W^{\alpha, p}(\mathbb{R}^n)$ coincides with $\left\{f \in L^p(\mathbb{R}^n);\ \|s_\alpha(f)\|_{L^p(\mathbb{R}^n)} < \infty\right\}$, where

$$s_\alpha(f)(x) := \left\{\int_0^\infty \left[\fint_{B(x,t)} |f(x) - f(y)|\, dy\right]^2 \frac{dt}{t^{1+2\alpha}}\right\}^{1/2}, \quad x \in \mathbb{R}^n.$$

- $B(x, t) := \{y \in \mathbb{R}^n : \ |y - x| < t\}$, $\fint_{B(x,t)} = \frac{1}{|B(x,t)|}\int_{B(x,t)}$
- [GKZ13] Not true when $\alpha \geq 1$:

  If $\alpha \geq 1$, then $f \in L^p(\mathbb{R}^n) + \|s_\alpha(f)\|_{L^p(\mathbb{R}^n)} < \infty \Rightarrow f \equiv$ Constant

[GKZ13] **A. Gogatishvili, P. Koskela and Y. Zhou**, *Characterizations of Besov and Triebel-Lizorkin spaces on metric measure spaces*, Forum Math. 25 (2013), 787-819.

Alabern-Mateu-Verdera characterization

- Define

$$S_\alpha(f)(x) := \left\{ \int_0^\infty \left| \fint_{B(x,t)} [f(x) - f(y)]\, dy \right|^2 \frac{dt}{t^{1+2\alpha}} \right\}^{1/2}.$$

Theorem ([AMV12])

Let $\alpha \in (0, 2)$ and $p \in (1, \infty)$. Then $f \in W^{\alpha, p}(\mathbb{R}^n)$ if and only if $f \in L^p(\mathbb{R}^n)$ and $S_\alpha(f) \in L^p(\mathbb{R}^n)$.

- Difference: $|f(x) - f(y)|$ in s_α is replaced by $f(x) - f(y)$ in S_α.
- $S_\alpha(f)$ does not depends on the differential structure of $\mathbb{R}^n$

[AMV12] **R. Alabern, J. Mateu and J. Verdera**, *A new characterization of Sobolev spaces on Rn*, Math. Ann. 354 (2012), 589-626.

- Key Point:
  - The Taylor expansion for $f \in C^2(\mathbb{R}^n)$ of order 2:
    $$f(y) = f(x) + \nabla f(x) \cdot (x-y) + O(|x-y|^2),\ x, y \in \mathbb{R}^n;$$
  - $⨏_{B(x,t)} (x-y)\, dy = 0$;
  - $$⨏_{B(x,\, t)} [f(x) - f(y)]\, dy = O(t^2),\ x \in \mathbb{R}^n.$$
- S_α provides smoothness up to order 2, but s_α only 1.
- This observation was originally used by Wheeden in 1969 to study Lipschitz (Besov) spaces.

Another observation on the square function $S_\alpha(f)$

- Rewrite

$$\fint_{B(x,t)} [f(x) - f(y)]\, dy = f(x) - B_t f(x)$$

with

$$B_t f(x) := \frac{1}{|B(x,t)|} \int_{B(x,t)} f(y)\, dy.$$

Then, Alabern-Mateu-Verdera's square function

$$S_\alpha(f)(x) := \left\{ \int_0^\infty \left| \fint_{B(x,\,t)} [f(x) - f(y)]\, dy \right|^2 \frac{dt}{t^{1+2\alpha}} \right\}^{1/2}, \quad \alpha \in (0, 2),$$

can be reformulated as

$$\left\{ \int_0^\infty \left| \frac{f(x) - B_t f(x)}{t^\alpha} \right|^2 \frac{dt}{t} \right\}^{\frac{1}{2}}.$$

Theorem [DGYY1]

Let $p \in (1, \infty)$. The following statements are equivalent:

(i) $f \in W^{2,p}(\mathbb{R}^n)$;

(ii) $f \in L^p(\mathbb{R}^n)$ and $\exists\, g \in L^p(\mathbb{R}^n)$ such that

$$\lim_{t\to 0^+} \frac{f - B_t f}{t^2} = g \quad \text{in} \;\; \mathcal{S}'(\mathbb{R}^n);$$

(iii) $f \in L^p(\mathbb{R}^n)$ and $\exists\, g \in L^p(\mathbb{R}^n)$ such that, for all $t \in (0, \infty)$ and a. e. $x \in \mathbb{R}^n$,

$$|f(x) - B_t f(x)| \le t^2 g(x).$$

• Key tool: $\lim_{t\to 0^+} \frac{\varphi - B_t\varphi}{t^2} = -\frac{1}{2(n+2)}\Delta\varphi$ in $\mathcal{S}(\mathbb{R}^n)$

[DGYY1] **F. Dai, A. Gogatishvili, D. Yang and W. Yuan**, *Characterizations of Sobolev spaces via averages on balls*, Nonlinear Anal. 128 (2015), 86-99.

Higher order case

- Observe that, for all $t \in (0, \infty)$ and $x \in \mathbb{R}^n$,

$$
\begin{aligned}
f(x) - B_t f(x) &= f(x) - \frac{1}{2} \fint_{B(0,1)} [f(x+ty) + f(x-ty)]\, dy \\
&= -\frac{1}{2} \fint_{B(0,1)} \left[f(x+ty) + f(x-ty) - 2f(x) \right] dy \\
&= -\frac{1}{2} \fint_{B(0,1)} \Delta_{ty}^2 f(x)\, dy.
\end{aligned}
$$

- $\Delta_h^1 f(x) = f(x+h) - f(x)$, $\Delta_h^M = \Delta_h^1(\Delta_h^{M-1})$

Thus, it is very natural to introduce a higher order average operator $B_{t,\ell}$ via the identity

$$f(x) - B_{\ell,t} f(x) = \frac{1}{C_\ell} \fint_{B(0,1)} \Delta_{ty}^{2\ell} f(x)\, dy, \quad x \in \mathbb{R}^n,\ \ell \in \mathbb{N}.$$

for some constant C_ℓ.

Observe that

$$\fint_{B(0,1)} \Delta_{ty}^{2\ell} f(x)\,dy = \sum_{k=0}^{2\ell} (-1)^k \binom{2\ell}{k} \fint_{B(0,1)} f(x+(\ell-k)ty)\,dy$$
$$= (-1)^\ell \binom{2\ell}{\ell} f(x) + 2(-1)^\ell \sum_{j=1}^{\ell} (-1)^j \binom{2\ell}{\ell-j} B_{jt} f(x).$$

By taking $C_\ell := (-1)^{\ell+1}\binom{2\ell}{\ell}$, one see that

$$B_{\ell,t} f(x) := -\frac{2}{\binom{2\ell}{\ell}} \sum_{j=1}^{\ell} (-1)^j \binom{2\ell}{\ell-j} B_{jt} f(x), \qquad t \in (0,\infty), \quad x \in \mathbb{R}^n.$$

$*$ Comparing with differences, $B_{\ell,t} f$ can be easily defined on metric measure spaces

$*$ $f(x) - B_{\ell,t} f(x) = O(t^{2\ell}), \quad \ell \in \mathbb{N}.$

Theorem [DGYY1]

Let $p \in (1, \infty)$ and $\ell \in \mathbb{N}$. The following statements are equivalent:

(i) $f \in W^{2\ell,p}(\mathbb{R}^n)$;

(ii) $f \in L^p(\mathbb{R}^n)$ and $\exists\, g \in L^p(\mathbb{R}^n)$ such that

$$\lim_{t\to 0^+} \frac{f - B_{\ell,t} f}{t^{2\ell}} = g \quad \text{in } \mathcal{S}'(\mathbb{R}^n);$$

(iii) $f \in L^p(\mathbb{R}^n)$ and $\exists\, g \in L^p(\mathbb{R}^n)$ such that, for all $t \in (0, \infty)$ and a. e. $x \in \mathbb{R}^n$,

$$|f(x) - B_{\ell,t} f(x)| \le t^{2\ell} g(x).$$

- How about Triebel-Lizorkin spaces and Besov spaces?

Theorem [DGYY1]

Let $p \in (1, \infty)$ and $\ell \in \mathbb{N}$. The following statements are equivalent:

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(iii) $f \in L^p(\mathbb{R}^n)$ and $\exists\, g \in L^p(\mathbb{R}^n)$ such that, for all $t \in (0, \infty)$ and a. e. $x \in \mathbb{R}^n$,

$$|f(x) - B_{\ell,t} f(x)| \le t^{2\ell} g(x).$$

- How about Triebel-Lizorkin spaces and Besov spaces?

4. Characterize Besov-Triebel-Lizorkin spaces via ball averages

Littlewood-Paley characterizations

$*$ idea: use $\{f - B_{\ell,2^{-k}} f\}_k$ to replace $\{\varphi_k * f\}_k$

Theorem [DGYY2]

Let $\ell \in \mathbb{N}$, $\alpha \in (0, 2\ell)$ and $q \in (0, \infty]$. Then $f \in B^{\alpha}_{p,q}(\mathbb{R}^n)$ if and only if $f \in L^p(\mathbb{R}^n)$ when $p \in (1, \infty)$ or $f \in C^0(\mathbb{R}^n)$ when $p = \infty$, and

$$|||f|||_{B^{\alpha}_{p,q}(\mathbb{R}^n)} := \|f\|_{L^p(\mathbb{R}^n)} + \left\{\sum_{k=1}^{\infty} 2^{k\alpha q} \|f - B_{\ell,2^{-k}} f\|^q_{L^p(\mathbb{R}^n)}\right\}^{1/q} < \infty.$$

Moreover, $||| \cdot |||_{B^{\alpha}_{p,q}(\mathbb{R}^n)}$ is equivalent to $\| \cdot \|_{B^{\alpha}_{p,q}(\mathbb{R}^n)}$.

[DGYY2] **F. Dai, A. Gogatishvili, D. Yang and W. Yuan**, *Characterizations of Besov and Triebel-Lizorkin spaces via averages on balls*, J. Math. Anal. Appl. 433 (2016), 1350-1368.

Theorem [YYZ,DGYY2]

Let $\ell \in \mathbb{N}$, $\alpha \in (0, 2\ell)$, $p \in (1, \infty)$ and $q \in (1, \infty]$. Then $f \in F^{\alpha}_{p,q}(\mathbb{R}^n)$ if and only if $f \in L^p(\mathbb{R}^n)$ and

$$|||f|||_{F^{\alpha}_{p,q}(\mathbb{R}^n)} := \|f\|_{L^p(\mathbb{R}^n)} + \left\| \left\{ \sum_{k=1}^{\infty} 2^{k\alpha q} |f - B_{\ell,2^{-k}} f|^q \right\}^{1/q} \right\|_{L^p(\mathbb{R}^n)} .$$

Moreover, $||| \cdot |||_{F^{\alpha}_{p,q}(\mathbb{R}^n)}$ is equivalent to $\| \cdot \|_{F^{\alpha}_{p,q}(\mathbb{R}^n)}$.

[YYZ] **D. Yang, W. Yuan and Y. Zhou**, *A new characterization of Triebel-Lizorkin spaces on Rn*, Publ. Mat. 57(2013), 57-82.

More on Littlewood-Paley characterizations

Z. He, D. Yang and W. Yuan, *Littlewood-Paley characterizations of second-order Sobolev spaces via averages on balls*, Canad. Math. Bull. 59(2016), 104-118.

Z. He, D. Yang and W. Yuan, *Littlewood-Paley characterizations of higher-order Sobolev spaces via averages on balls*, Math. Nachr. 291(2018), 284-325.

D.-C. Chang, J. Liu, D. Yang and W. Yuan, *Littlewood-Paley characterizations of Hajłasz-Sobolev and Triebel-Lizorkin spaces via averages on balls*, Potential Anal. 46(2017), 227-259.

Hajłasz type gradients via ball averages

Definition [YY17]

Let $\alpha \in (0, \infty)$ and $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. A sequence $\vec{g} := \{g_j\}_{j \geq 0}$ of non-negative measurable functions is called a (α, ℓ)-order Hajłasz type gradient sequence of f if, for each j, there exists a set $E_j \subset \mathbb{R}^n$ with measure zero such that

$$|f(x) - B_{\ell, 2^{-j}} f(x)| \leq 2^{-j\alpha} g_j(x), \quad x \in \mathbb{R}^n \setminus E_j. \tag{1}$$

Each g_j satisfying (1) is called an (α, ℓ)-order Hajłasz type gradient of f at level j.

[YY17] **D. Yang and W. Yuan**, *Pointwise characterizations of Besov and Triebel-Lizorkin spaces in terms of averages on balls*, Trans. Amer. Math. Soc. 369(2017), 7631-7655.

Pointwise characterizations of B-space

Theorem [YY17]

Let $\ell \in \mathbb{N}$, $\alpha \in (0, 2\ell)$, $p \in (1, \infty]$ and $q \in (0, \infty]$. Then $f \in B^{\alpha}_{p,q}(\mathbb{R}^n)$ if and only if $f \in L^p(\mathbb{R}^n)$ when $p \in (0, \infty)$ or in $C^0(\mathbb{R}^n)$ when $p = \infty$, and there exists a (α, ℓ)-order Hajłasz type gradient sequence $\vec{g} = \{g_j\}_{j \geq 0}$ of f such that

$$\inf \left\{ \sum_{k \geq 0} 2^{k\alpha q} \|g_k\|_{L^p(\mathbb{R}^n)} \right\}^{1/q} < \infty,$$

where the infimum is taken over all such $\vec{g} = \{g_j\}_{j \geq 0}$.

Pointwise characterizations of F-space

Theorem [YY17]

Let $\ell \in \mathbb{N}$, $\alpha \in (0, 2\ell)$ and $p \in (1, \infty)$, $q \in (1, \infty]$. Then $f \in F^{\alpha}_{p,q}(\mathbb{R}^n)$ if and only if $f \in L^p(\mathbb{R}^n)$ and there exists a (α, ℓ)-order Hajłasz type gradient sequence $\vec{g} = \{g_j\}_{j\geq 0}$ of f such that

$$\inf \left\| \left\{ \sum_{k\geq 0} 2^{k\alpha q} |g_k|^q \right\}^{1/q} \right\|_{L^p(\mathbb{R}^n)} < \infty,$$

where the infimum is taken over all such $\vec{g} = \{g_j\}_{j\geq 0}$.

- Further applications of these ball average characterizations ??

Thank you for your attention!

Pointwise characterizations of F-space

Theorem [YY17]

Let $\ell \in \mathbb{N}$, $\alpha \in (0, 2\ell)$ and $p \in (1, \infty)$, $q \in (1, \infty]$. Then $f \in F^{\alpha}_{p,q}(\mathbb{R}^n)$ if and only if $f \in L^p(\mathbb{R}^n)$ and there exists a (α, ℓ)-order Hajłasz type gradient sequence $\vec{g} = \{g_j\}_{j\geq 0}$ of f such that

$$\inf \left\| \left\{ \sum_{k\geq 0} 2^{k\alpha q} |g_k|^q \right\}^{1/q} \right\|_{L^p(\mathbb{R}^n)} < \infty,$$

where the infimum is taken over all such $\vec{g} = \{g_j\}_{j\geq 0}$.

- Further applications of these ball average characterizations ??

Thank you for your attention!